Essentializing Equilibrium Concepts*

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Abstract

Given an (extensive) game form, an equilibrium concept and a (behavior) strategy profile, what parts of the game form are never relevant to check if the strategy profile is (essentially) an equilibrium of a game with that game form? What is left after removing the irrelevant parts is what we call *essential*. In this paper we *essentialize* a wide variety of equilibrium concepts and present several applications of our analysis.

What do we mean by "essentialize"?

In this paper we present a framework that allows to identify parts of a game that may be dispensed with to check whether a certain outcome is an equilibrium outcome or not.

Many different equilibrium concepts have been studied in the game theoretical literature; van Damme (1991) provides an introduction to this field. In this paper we analyze whether the most important ones can be *essentialized*. Given a game, most equilibrium concepts select those strategy profiles that satisfy a certain condition in the whole game tree. However, if we want to study a specific strategy profile, there are many branches of the game tree that are irrelevant. Informally, to essentialize an equilibrium concept is to find, for each strategy profile, the regions of the game tree that are needed to check whether the profile (essentially) satisfies the requirements of the equilibrium concept.

More specifically, consider an equilibrium concept for extensive games, for instance, subgame perfect equilibrium, hereafter SPE. Quite generally, to know whether a strategy profile is a SPE of a given game, there are payoffs to which we do not need to look at: for the game G and strategy profile b in the left part of Figure 1, the payoff $(2,2)$ is irrelevant to check if b is a SPE of G or not (it can never be reached after unilateral deviations). More generally, consider the game G' in the right part of Figure 1. That is, the payoff $(2,2)$ in G has been replaced with a matching pennies game. In this case, to check if b' is a SPE, the behavior in the subgame

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matters. Indeed, b' is not a SPE of G' because it is not a Nash equilibrium of the matching pennies subgame. Yet, we can ask the following question: is the outcome of b' a SPE outcome? or, equivalently, is any SPE of G' realization equivalent to b'? To answer this question, the payoffs and also the behavior in the matching pennies game are completely irrelevant (because this subgame cannot be reached via unilateral deviations from b'). Throughout this paper, we think of a game (G) as a game form (Γ) together with a payoff function. Then, for every game whose game form coincides with that of G', the payoffs and behavior in the proper subgame of the game are irrelevant to know if the outcome of b' is a SPE outcome.

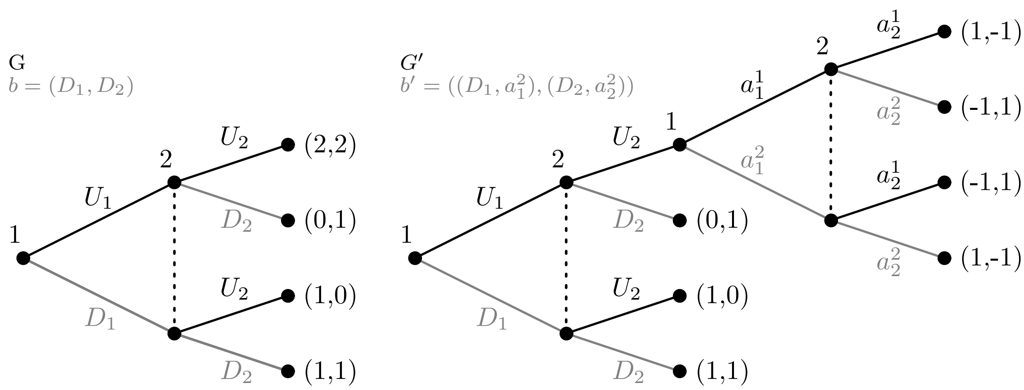


Figure 1: Games G and G' and strategies b and b' (in gray).

The above discussion suggests that we may confirm that the outcome of a given strategy profile is an equilibrium outcome without taking into account all the parts of the game tree, *i.e.*, there is some information about the game that can be disregarded and still be in good shape to certify that the outcome at hand is an equilibrium outcome. Thus, identifying these "irrelevant" parts of the tree might be useful. With this motivation in mind, we devote this paper to study the following problem:

> *"Given an equilibrium concept EC, an (extensive) game form* Γ*, and a (behavior) strategy profile* b*, identify* W*, a minimal collection of information sets of* Γ*, with the following property:*
> *If a game* G *has game form* Γ *and* b *is an equilibrium of* G*, then, whatever changes are made in the payoffs and strategies outside* W*, the outcome of* b *will be an equilibrium outcome in the resulting game."*

Games in Figure 1 suggest that not being in W, *i.e.*, being irrelevant, is related to the fact of not being reachable through (sequences of) unilateral deviations. Even though this is the case, it is hard to find an appropriate mathematical formulation of the above problem that is operative for all equilibrium concepts. Under our approach, given an equilibrium concept, a game form, and a strategy profile, there is a unique minimal collection of information sets satisfying the above property. We refer to it as the *essential* collection for EC, Γ, and b. Then, we characterize the essential collections that arise from different equilibrium concepts.

We consider that the main contribution of this paper is to provide a definition of essential collection that is useful in the various ways we describe below.

A first application of our results runs as follows. Given an equilibrium concept, a game, and strategy profile b, we provide a reduced game such that, if the reduced version of b is an equilibrium of the reduced game, then the outcome of b is an equilibrium outcome in the original game. Since the reduced game might be notably smaller than the original game, the above verification might be much easier. Indeed, a very similar approach to that underlying the reduced game notion has already been used for SPE in Osborne (1993) in a model of political competition. Section 2, we use Osborne's model to illustrate the different implications and applications of our results.

The results obtained for the reduced game also allow to get a better understanding of the structural robustness of the different equilibrium concepts. More precisely, we can compare how robust each equilibrium concept is to modifications in the game such as changes in the sets of strategies, in the players of the game, in the information available to the players, and also in the payoffs; this kind of robustness checks have already been made in Kalai (2005, 2006) for Nash equilibrium in the so called large games. Moreover, we show that, even in situations in which the underlying game is only partially-specified, we can sometimes know whether a given outcome is an equilibrium outcome of any game fulfilling the partial specifications we have.

The other main application of our analysis is what we call *virtual equilibrium concepts*. For each equilibrium concept we define its virtual version by dropping the restrictions on the behavior in the "irrelevant" parts of the game tree. We show that, given an equilibrium concept, as far as the original game has some equilibrium, the sets of equilibrium outcomes and virtual equilibrium outcomes coincide. Yet, there can be games in which the set of virtual equilibria is nonempty whereas there is no non-virtual equilibrium; the virtual equilibria being still sensible in the spirit of its non-virtual counterpart. A similar approach to that of virtual equilibria has been independently taken in Groenert (2007), where the author introduces the idea of *trimmed equilibrium* and applies it to subgame perfect equilibrium and weak perfect Bayesian equilibrium; nonetheless, our approach is more general since it allows to define virtual versions of a wide variety of equilibrium concepts and, moreover, virtual equilibria are just an application of our main contribution: the essential collections. In García-Jurado and González-Díaz (2006), the authors use the virtual version of subgame perfect equilibrium to get a folk theorem for a class of repeated games in which the existence of subgame perfect equilibria is not guaranteed. Also, the equilibrium notion used in Osborne (1993) is very close to the virtual version of subgame perfect equilibrium.

To some extent, the main contribution of this paper is to provide the framework to develop for many equilibrium concepts the kind of analysis partially developed for SPE in Osborne (1993) and García-Jurado and González-Díaz (2006).

The paper is structured as follows. In Section 1 we introduce the basic notations and also define the main concepts to be analyzed. In Section 2 we present an overview of the main results of the paper and build upon the model in Osborne (1993) to illustrate some of their implications and applications. In Sections 3 and 4 we characterize the essential collections for a wide variety of solution concepts, including the most widely used ones. In Section 5 we present some applications of our analysis.

1 Formal definitions

1.1 Basic notations

We develop our analysis for finite extensive games with perfect recall.We follow the representation of an extensive game given in Fudenberg and Tirole (1991a).[1] We denote an *(extensive) game form* by Γ and it is characterized by i) a finite game tree with root $r(\Gamma)$, ii) a finite *set of players* $N = \{1, \dots, n\}$, iii) the sets of *nodes*, *terminal nodes*, and *information sets* of Γ, denoted by $X(\Gamma)$, $Z(\Gamma)$, and $U(\Gamma)$, respectively, and iv) the probabilities of nature choices, if any. Under this representation, nature only moves at $r(\Gamma)$ (once and for all). As in Fudenberg and Tirole (1991a), we think of $U(\Gamma)$ as a partition of $X(\Gamma)$, *i.e.*, each terminal node is also an information set. We refer to the subsets of $U(\Gamma)$ as *collections* (of information sets). Also, we use $U_i(\Gamma)$ to denote the information sets belonging to a player $i \in N$.

A *game (in extensive form)* is a pair $G = (\Gamma, h)$ where Γ is a game form and $h\colon Z(\Gamma) \to \mathbb{R}^n$ is the *payoff function*, *i.e.*, $h(z) = (h_1(z), \dots, h_n(z))$, where $h_i(z)$ denotes the payoff received by i if z is realized. We denote by $\mathcal{G}(\Gamma)$ the set of games with game form Γ. Given a game G (or a game form Γ), $B(\Gamma) = \prod_{i=1}^{n} B_i(\Gamma)$ denotes the set of *behavior strategy profiles.* Given $b \in B(\Gamma)$, we slightly abuse notation and use $h_i(b)$ to denote the (expected) payoff to player i when b is played. Given $G \in \mathcal{G}(\Gamma)$, let $M_G := \max_{i \in N, z \in Z(\Gamma)} |h_i(z)| + 1$.

Let Γ be a game form. Let $i \in N$ and let $b, \bar{b} \in B(\Gamma)$. We say that b and $\bar{b}$ are *realization equivalent* if all the nodes of Γ are reached with the same probabilities under b and $\bar{b}$. Given $b \in B(\Gamma)$, $\pi(b)$, denotes the collection of information sets that are reached with positive probability when b is played, *i.e.*, $\pi(b)$ can be seen as the union of all paths of play that might be realized when b is played. Hence, we slightly abuse language and refer to $\pi(b)$ itself as the *path* of b.

The node x is a *predecessor* of node y, denoted by $x \prec y$, if $x \neq y$ and x is in the path from the root to y; $x \preceq y$ means that either $x \prec y$ or $x = y$. If $x \preceq y$, then the *path of nodes from x to y* is the sequence formed by x, y, and the nodes in between x and y. Similarly, $u \in U(\Gamma)$ is a *predecessor* of $v \in U(\Gamma)$, denoted by $u \prec v$, if $u \neq v$ and there are $x \in u$ and $y \in v$ such that $x \prec y$; $u \preceq v$ means that either $u \prec v$ or $u = v$.[2] If $x \preceq y$, then the *path of information sets from x to y* is the sequence formed by u_x, u_y, and the information sets containing nodes in between x and y. Whenever we represent a path of nodes or information sets as a sequence $\{x^1, \dots, x^k\}$ it is implicitly assumed that $x^1 \prec x^2 \prec \dots \prec x^k$. Also, given $x \in X(\Gamma)$ and $u \in U(\Gamma)$, $x \prec u$ and $u \prec x$ are defined in the obvious manner.

Given a collection $W \subset U(\Gamma)$, we say that W is *closed* (under $\preceq$) if, for each $v \in W$ and each $u \in U(\Gamma)$, $u \prec v$ implies that $u \in W$. The smallest closed collection containing a collection W is denoted by $\langle W \rangle$.[3] Given a collection $W \subset U(\Gamma)$, we say that W is *terminal* if, for each $u \in W$ and each $x \in u$, there is $z \in W \cap Z(\Gamma)$ such that $x \preceq z$. Arbitrary unions and intersections of closed collections lead to closed collections; also, arbitrary unions of terminal collections lead to terminal collections.

[1] This representation is equivalent to the classic one given by Kuhn (1953) and further developed in Selten (1975) and Kreps and Wilson (1982).

[2] Note that it is possible to have both $u \prec v$ and $v \prec u$.

[3] More formally, let $\rho\colon U(\Gamma) \to 2^{U(\Gamma)}$ be the operation defined by $\rho(v) := \{u \in U(\Gamma) : u \preceq v\}$. A set $W \subset U(\Gamma)$ is closed under ρ if, for each $v \in W$, $\rho(v) \subset W$. Now, $\langle W \rangle$ denotes the *closure* of W under the operation ρ, *i.e.*, the smallest closed subset of $U(\Gamma)$ containing W. Then, W is closed under $\preceq$ if $\langle W \rangle = W$.

Lemma 1. *Let Γ be a game form. Let W and $\bar{W}$ be two collections in $U(\Gamma)$ closed under $\preceq$. If $\bar{W}$ is terminal and $\bar{W}\backslash W \neq \emptyset$, then $(\bar{W}\backslash W) \cap Z(\Gamma) \neq \emptyset$.*

Proof. Let $u \in \bar{W}\backslash W$. Since $\bar{W}$ is terminal, there is $z \in \bar{W} \cap Z(\Gamma)$ such that $u \preceq z$. Now, since W is closed under $\preceq$, $u \notin W$, and $u \preceq z$, we have that $z \notin W$. □

Given $b \in B(\Gamma)$ and $W \subset U(\Gamma)$, b_W denotes the restriction of b to the information sets in W; similarly, b_{-W} denotes the restriction of b to the information sets outside W. Take a pair of games $G = (\Gamma, h), \bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ and a collection $W \subset U(\Gamma)$. The *W-combination* of G and $\bar{G}$ is defined by $G \otimes_W \bar{G} := (\Gamma, h \otimes_W \bar{h})$, where $h \otimes_W \bar{h}$ coincides with h in $W \cap Z(\Gamma)$ and with $\bar{h}$ in $Z(\Gamma) \setminus W$. Thus, $\otimes_W$ is not commutative. Similarly, given $b, \bar{b} \in B(\Gamma)$, $b \otimes_W \bar{b} := (b_W, \bar{b}_{-W})$, *i.e.*, the profile that consists of playing according to b in W and to $\bar{b}$ anywhere else. For the sake of notation, when no confusion arises, we use the abbreviated notations $G^{\otimes}$, $h^{\otimes}$, and $b^{\otimes}$.

The equilibrium concepts we explicitly discuss in this paper are *sequential rationality* (SR), Nash equilibrium (NE), subgame perfect equilibrium (SPE), weak perfect Bayesian equilibrium (WPBE), sequential equilibrium (SE), and perfect equilibrium (PE). Also, in Section 4 we present an analysis that carries out for a family of equilibrium concepts that range from sequential rationality to several versions of perfect Bayesian equilibrium. Since every finite extensive game has a perfect equilibrium in behavior strategies, existence is ensured for all the equilibrium concepts we discuss. Interestingly, in Section 5 we discuss a direct application of our analysis to situations in which existence is not guaranteed (restriction to pure strategies, infinite sets of strategies,...).

1.2 Essential collections

Definition 1. Fix an equilibrium concept EC. Let Γ be a game form and $b \in B(\Gamma)$. A collection $W \subset U(\Gamma)$ is *sufficient* for EC, Γ, and b if it has the following properties:

i) $\pi(b) \subset W$, *i.e.*, W contains the path of b.

ii) Let $G, \bar{G} \in \mathcal{G}(\Gamma)$ be such that $b \in \text{EC}(G)$ and $\text{EC}(\bar{G}) \neq \emptyset$. Then, there is $\hat{b} \in \text{EC}(G \otimes_W \bar{G})$ such that b and $\hat{b}$ coincide in W.

Note that i) and ii) together imply that b and $\hat{b}$ are realization equivalent. In words, the idea is the following: take a collection W that is sufficient (for EC, Γ, and b) and take $G \in \mathcal{G}(\Gamma)$ for which b is an equilibrium. Now, if we change the payoffs outside W, provided that this new game has some equilibrium, then there will be one that is realization equivalent to b.

Consider again the game G' introduced during the motivation (Figure 1), with SPE as the equilibrium concept. In this case, regardless of the payoffs we put instead of those of the matching pennies subgame, there will always be a SPE of the game that is realization equivalent to b'. As we will see below, this is because the collection that is left after removing the matching pennies subgame is sufficient for SPE, Γ, and b'. Indeed, if we let W be the above collection, to answer the question "does any equilibrium of G' coincide with b' in W?" the behavior outside W does not matter. If the answer is positive then, regardless of how we change the payoffs outside W, the answer will remain positive for the new game; in particular, the outcome of b will be an equilibrium outcome in the new game.

Note that the property of being a sufficient collection only depends on the equilibrium concept at hand and the given game form and strategy profile. That is, it does not depend on the possible payoffs we might associate with the game form.

The gist of being a sufficient collection is contained in property ii). Hence, one might argue about the necessity of i). Some minimality requirement needs to be imposed on a sufficient collection, since an empty collection always satisfies ii). Thus, the path of b is a natural candidate since we then ensure that $(b_W, \hat{b}_{-W})$ is realization equivalent to b, which was an important element in the motivation section.

Lemma 2. *If W is sufficient for EC, Γ and b, then it is also sufficient for any other $\bar{b}$ such that $b_W = \bar{b}_W$.*

Proof. Straightforward. □

Lemma 3. *The intersection of sufficient collections is a sufficient collection.*

Proof. Fix an equilibrium concept EC. Let Γ be a game form and $b \in B(\Gamma)$. Let W and $\bar{W}$ be two sufficient collections (for EC, Γ, and b). First, $W \cap \bar{W}$ contains $\pi(b)$. Then, let G and $\bar{G} \in \mathcal{G}(\Gamma)$ be such that $b \in \mathrm{EC}(G)$ and $\mathrm{EC}(\bar{G}) \neq \emptyset$. We want to find $\hat{b} \in \mathrm{EC}(G \otimes_{W \cap \bar{W}} \bar{G})$ such that b and $\hat{b}$ coincide in $W \cap \bar{W}$. Since W is a sufficient collection, there is $\tilde{b} \in \mathrm{EC}(G \otimes_W \bar{G})$ that coincides with b in W. Let $\tilde{G} = G \otimes_W \bar{G}$. Since $\bar{W}$ is a sufficient collection, there is $\hat{b} \in \mathrm{EC}(\tilde{G} \otimes_{\bar{W}} \bar{G})$ that coincides with $\tilde{b}$ in $\bar{W}$. Now, by definition, $\hat{b}$ coincides with b in $W \cap \bar{W}$ and $\tilde{G} \otimes_{\bar{W}} \bar{G} = G \otimes_{W \cap \bar{W}} \bar{G}$. □

Corollary 1. *Fix an equilibrium concept* EC. *Let Γ be a game form and $b \in B(\Gamma)$. Then, there is a unique minimal collection that is sufficient for* EC, Γ, *and b. Moreover, there is a unique minimal collection that is closed and sufficient for* EC, Γ, *and b.*

Proof. Take the intersection of all the sufficient collections for EC, Γ, and b. Since Γ is always a sufficient collection and all the sufficient collections contain $\pi(b)$, non-emptiness is guaranteed. The above intersection is contained in all the sufficient collections and its sufficiency follows from Lemma 3. The proof of the second statement is analogous, since Γ is a closed collection and the intersection of closed collections is a closed collection. □

Remark 1. Note that if W and $\bar{W}$ are two collections such that $W \subset \bar{W}$ and W is sufficient (for some EC, Γ, and b), then it need not be the case that $\bar{W}$ is also sufficient. The reason is that the condition that b and $\hat{b}$ coincide in $\bar{W}$ (Definition 1) can be much more demanding than the corresponding condition for W.

Definition 2. Fix an equilibrium concept EC. Let Γ be a game form and $b \in B(\Gamma)$. The *essential* collection for EC, Γ, and b, denoted by $W_{\mathrm{EC}}(\Gamma, b)$, is defined as the unique minimal collection that is closed under $\preceq$ and sufficient for EC, Γ, and b.

To *essentialize* an equilibrium concept EC is to find the map W_{EC} that assigns, to each pair (Γ, b), the essential collection $W_{\mathrm{EC}}(\Gamma, b)$.

Some explanation is needed for the requirement that an essential collection has to be closed. First, it is quite natural. Think, for instance, of a belief-based equilibrium concept. In this case, the closedness under $\preceq$ says that, if an information set $u \in U_i(\Gamma)$ is in the essential collection, *i.e.*, player i's behavior at u is relevant for EC, Γ, and b, then what b prescribes for information sets that precede u should also be relevant, as it might affect the beliefs and behavior of i at u. Again, it might be argued that this should be a consequence of the definition and not part of the definition itself. Nonetheless, if this requirement is removed, then some unnatural essential collections might appear. Second, the closedness requirement allows for a more streamlined analysis and more natural constructions for the essential collections associated with the different equilibrium concepts. Refer to Appendix A for further arguments for and against this requirement.

2 Discussion of the contribution

In the previous section we formally defined what we mean by essentialize an equilibrium concept. As we will see in the forthcoming sections, our definition is general enough to be applied to all the classic equilibrium concepts. Unfortunately, the price of this generality is that the analysis becomes quite cumbersome already for Nash equilibrium. An important part of the paper is to formally characterize the essential collections associated with the different equilibrium concepts. Since this comprehensive exercise is quite arid, we present in this section an informal overview of the main results of the paper and discuss the relevance of our contribution. For the sake of exposition we abstract from the fact that essential collections have to be closed. For the precise characterizations, refer to Sections 3 and 4.

We divide the equilibrium concepts to characterize in two big groups: non-belief-based equilibrium concepts (NE, SPE, and PE) and belief-based equilibrium concepts (SR, WPBE, SE, and a whole family of intermediate equilibrium concepts).

The characterizations for the non-beliefs-based equilibrium concepts are quite intuitive and barely bring any new insights relative to the nature of the different equilibria. Informally, this characterizations say the following. Let Γ be a game form and $b \in B(\Gamma)$, then,

Nash equilibrium: The essential collection consists of all the information sets that can be reached after a unilateral deviation from b.

Subgame perfect equilibrium: A subgame is relevant if it can be reached through a series of unilateral deviations from b at other subgames. An information set belongs to the essential collection if it can be reached after a unilateral deviation from b at a relevant subgame. That is, the essential collection for SPE can be easily constructed iteratively: at a given step, we add those subgames that can be reached after deviations from b at subgames reached in the previous steps.

Perfect equilibrium: Every information set belongs to the essential collection.

In particular, in a game with perfect information, since a subgame begins at every node, the essential collection for SPE contains all the nodes of the game, *i.e.*, it coincides with that of PE. Yet, this is not the case for NE.

Note that, for the above characterizations, the more demanding an equilibrium concept is, the larger its corresponding essential collections are. This result is very natural and one could expect the same relations to hold for belief-based equilibrium concepts. Remarkably, not only these relations do not hold, but the opposite ones do, *i.e.*, the more demanding a belief-based equilibrium concept is, the smaller the corresponding essential collections are. More specifically,[4]

Sequential rationality: Every information set belongs to the essential collection.

Weak perfect Bayesian equilibrium: An information set u belongs to the essential collection if there is an assessment (b, μ) such that i) μ is calculated using Bayes rule in the path of b and ii) according to μ, a node in u is reached with positive probability with a series of unilateral deviations from b.

Sequential equilibrium: An information set u belongs to the essential collection if there is an assessment (b, μ) such that i) μ is consistent with b and ii) according to μ, a node in u is reached with positive probability with a series of unilateral deviations from b.

Moreover, our approach characterizes the essential collections associated with a family of belief-based equilibrium concepts in an analogous manner. The reader can already note the parallelism between the characterization of the essential collection for WPBE and that of SE above; when applied to SR this approach would say that an information set u belongs to the essential collection if there is an assessment (b, μ) such that, i) μ is any system of beliefs and ii) according to μ, a node in u is reached with positive probability with a sequence of unilateral deviations from b; and, clearly, with no restrictions on the beliefs, every node can always be reached after a series of unilateral deviations.

As written, the above characterizations for belief-based equilibrium concepts may also seem quite natural, but they imply that the more restrictive equilibrium concepts have smaller essential collections. For instance, for every game form and every strategy profile, the essential information sets for SE are a subset of those for WPBE or, equivalently, if an information set is irrelevant for WPBE, then it is irrelevant for SE as well; we show in the example below that the converse is not true in general. As a rough intuition for the latter implications, note that, when dealing with the belief-based equilibrium concepts above, the only difference in their definitions lies in the set of beliefs that can be considered; the less restrictive equilibrium concepts allow for more beliefs and hence, more parts of the game tree can be reached after (sequences of) unilateral deviations, which ultimately implies that the essential collections become larger for the less restrictive equilibrium concepts.

We present now an example to illustrate some implications of the above characterizations and also some applications of these and other results in the paper.

2.1 A candidate positioning game (Osborne, 1993)

In this example we discuss our contribution within the temporal model of political competition of Osborne (1993, Section 4); also, throughout the exposition we will relate the arguments used there with our approach. The relevant equilibrium concept will be SPE. The reader interested in a similar discussion of the implications our results for belief-based equilibrium concepts is referred to Appendix B, where we present another example with a deeper discussion that focuses on WPBE and SE.

For the sake of exposition, we present a slightly modified version of the original model, also omitting some elements that are not needed to illustrate our approach. The game has three players, which represent the three potential candidates in an election. There is a continuum of

[4] Refer to Section 4 for the definition of *assessment* and of the different belief-based equilibrium concepts.

voters, each of whom has a most preferred or ideal policy. Voter's ideal policies are given by the continuous distribution function F, whose support is the $[0,1]$ interval. Voters vote sincerely, *i.e.*, each voter endorses the candidate whose position is closest to his ideal; if indifferent, he decides randomly. Candidates just want to win the election by plurality rule (get more votes than any other candidate). At each period $t \in \{1, \ldots, T\}$ $(T > 2)$, candidates simultaneously decide whether to enter the competition or to wait. Candidate i enters the competition by announcing a policy p_i in the interval $[0,1]$. Policies are decided once and for all. Hence, at each period, a player who has already announced a policy cannot take any further action and, otherwise, he can either announce a policy, *i.e.*, a number in $[0,1]$, or decide to wait, which is denoted by w. Candidates can only use pure strategies.[5] A player who plays w in every period is just a player who decides to stay out of the election. Once period T is over, the election is held and the candidate with more votes wins. Let $\Gamma(3)$ denote this 3-player game.

As Osborne argues, "in $\Gamma(3)$, as in other sequential games in which some choices are made simultaneously, the spirit of subgame perfect equilibrium is captured by a notion that requires only a partial specification of the player's strategies" and the idea behind this observation is very close to our notion of essential collection. Suppose that we want to study a strategy profile in which players 1 and 2 enter in period 1 with policies p_1 and p_2, respectively, whereas player 3 chooses w in every period. Let b be a strategy profile in which the on-path behavior is the one we have just described.

Again, following Osborne (1993): to fully describe b, for player 1 we must "specify an action in period 2 for every first-period profile of actions (w, s_2, s_3), where s_2 and s_3 are members of $[0,1] \cup \{w\}$. However, there is just *one* relevant subgame in which Player 1 has to take an action: the one that follows the first-period action profile (w, x_2, w)".

Essential collections for SPE. Now, let W_{SPE} be the essential collection for SPE, Γ, and b. Then, following the informal characterization above, it is easy to see that the only information set of the form (w, s_2, s_3) that would belong to W_{SPE} would indeed be (w, x_2, w), since all the others involve a multilateral deviation at period 1. That is, an important advantage of our approach is that it helps to study if different outcomes of the game are equilibrium outcomes or not, since there is no need to check the incentives in many of the subgames of the game.

The reduced game. In Section 5, given a game G, we associate a reduced game G_W with each (closed) collection of information sets W; the basic idea is to remove from G all the information sets that are not in W in such a way that what is left still forms a game. For instance, when studying the strategy profile b, none of the subgames starting at information sets of the form (w, s_2, s_3) would be the root of a subgame in the reduced game (except for (w, x_2, w)). Now, (by Proposition 6) if the restriction of b to the reduced game is a SPE of the reduced game, then b is a SPE of $\Gamma(3)$ (provided that $\Gamma(3)$ indeed has at least one SPE).

Structural robustness. The main application of the reduced game may be to the study of the structural robustness of the different equilibrium concepts. Suppose that we already knew that $b \in \text{SPE}(\Gamma(3))$ but, how robust would this equilibrium be to structural changes in the game? Suppose that, in order to encourage early positioning of candidates, the following rule is imposed. If no candidate has entered the competition after period 2, then the election is suspended. Would b still be an equilibrium of the new game? Since no subgame at which the

[5] Osborne argues that, in this setting, "the problem of finding equilibria in mixed strategies seems intractable" and, moreover, "voters may have an aversion to candidates who choose their positions randomly...".

election is suspended belongs to the reduced game associated with b (they cannot be reached after unilateral deviations from b, where two candidates enter already in period 1), the above change in the rules of the game would have no impact for the profile b. That is, whether b is an equilibrium outcome or not is independent (robust) from those changes in the rules of $\Gamma(3)$ that only affect information sets outside the reduced game associated with b.

Partial-specifications of the game. This issue is very related to the one above. The idea is that essential collections may help to give some information about the equilibrium outcomes of games that are not completely specified. Suppose that, in $\Gamma(3)$, we have no idea about how the game unfolds if no player has entered the competition after period T. Even in this case we know (by Corollary 4) that, no matter how the game is defined from that point onwards, the outcome of b is going to be a SPE outcome. Hence, essential collections help to identify what misspecifications in the game are irrelevant for different strategies and equilibrium concepts.

Virtual equilibrium concepts. Suppose that there are some subgames of game $\Gamma(3)$ for which we do not even know whether a Nash equilibrium exists or not. Then, it might be that the game $\Gamma(3)$ has no SPE. This motivates the definition of virtual equilibrium concepts. We say that a strategy profile b is a virtual SPE if it is a SPE of the reduced game associated with its essential collection (for SPE and the game form at hand); and the virtual version of any other equilibrium concept is defined analogously. Hence, for the strategy b to be a virtual SPE we need that all the subgames of the corresponding reduced game have a Nash equilibrium, but we do not care about this for subgames outside the essential collection associated with b. Given a virtual equilibrium, we can always replace the non-equilibrium behavior outside the essential collection by equilibrium behavior (if this exists) to get an equilibrium in the classic sense. Then, (by Proposition 6) if the set of SPE of the original game is nonempty, the set of SPE outcomes and virtual SPE outcomes coincide (which justifies the name virtual).

Actually, the equilibrium notion introduced in Osborne (1993) is extremely close to the virtual version of SPE. Indeed, Osborne wrote "the advantage of working with this notion of equilibrium in the game $\Gamma(3)$ is that it is not necessary ... to worry about the existence of an equilibrium, in 'irrelevant subgames' " and "the relation between an equilibrium in this sense and a subgame perfect equilibrium is close: a subgame perfect equilibrium is an equilibrium and if every subgame has a subgame perfect equilibrium then an equilibrium is associated with at least one subgame perfect equilibrium", which is analogous to what we said above for virtual equilibrium concepts: every EC is a VEC and, if an EC exists, for each VEC we can find an EC with the same outcome.

3 Essentializing non-belief-based equilibrium concepts

We devote this section to the essentialization of the classic equilibrium concepts: Nash equilibrium, subgame perfect equilibrium, and perfect equilibrium. As we have already said, these characterizations barely bring new insights concerning these equilibrium concepts. Yet, there are some important reasons for also undergoing these characterizations. First, to some extent, the fact that we get intuitive results for these equilibrium concepts reassures the adequacy of our definitions. Second, to get the reader familiar with our approach and with the techniques of the proofs; all the characterizations in the paper share common ideas, but the proofs become more involved as we move on. Last but not least, for the sake of completeness.

We introduce now a stronger version of sufficiency that will be quite convenient to prove the characterization results below.

Definition 3. Fix an equilibrium concept EC. Let Γ be a game form and $b \in B(\Gamma)$. A collection $W \subset U(\Gamma)$ is strongly sufficient for EC, Γ, and b if it has the following properties:

i) $\pi(b) \subset W$, *i.e.*, W contains the path of b.

ii) Let $\bar{b} \in B(\Gamma)$ and $G, \bar{G} \in \mathcal{G}(\Gamma)$ be such that $b \in \mathrm{EC}(G)$ and $\bar{b} \in \mathrm{EC}(\bar{G})$. Then, $b \otimes_W \bar{b} \in \mathrm{EC}(G \otimes_W \bar{G})$.

3.1 Nash equilibrium

Let Γ be a game form and $b \in B(\Gamma)$. Then, let $W^b_{\mathrm{NE}} \subset U(\Gamma)$ be defined as the closure under $\preceq$ of the collection of information sets that can be reached after at most one unilateral deviation from b, *i.e.*, $W^b_{\mathrm{NE}} := \langle \{u \in U(\Gamma) \,:\, \text{there are } i \in N \text{ and } b'_i \in B_i(\Gamma) \text{ such that } u \in \pi(b_{-i}, b'_i)\}\rangle$. Note that $\pi(b) \subset W^b_{\mathrm{NE}}$ (just take $b'_i = b_i$) and W^b_{NE} is a terminal collection. Figure 2 illustrates the previous definition. Not surprisingly, the collection W^b_{NE} suffices to essentialize NE.

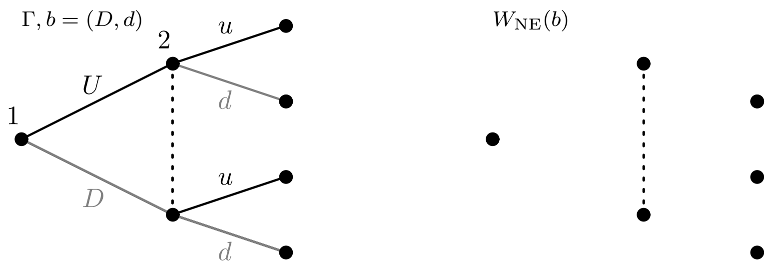


Figure 2: The collection W^b_{NE}.

Proposition 1. *W^b_{NE} is the essential collection for* NE, Γ*, and* b.

Proof. First, we show that W^b_{NE} is strongly sufficient for NE, Γ, and b. By definition, $\pi(b) \subset W^b_{\mathrm{NE}}$. Let $\bar{b} \in B(\Gamma)$ and $G = (\Gamma, h), \bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ be such that $b \in \mathrm{NE}(G)$ and $\bar{b} \in \mathrm{NE}(\bar{G})$. Suppose $b^{\otimes} \notin \mathrm{NE}(G^{\otimes})$. Then, there are $i \in N$ and $b'_i \in B_i(\Gamma)$ such that $h^{\otimes}_i(b^{\otimes}_{-i}, b'_i) > h^{\otimes}_i(b^{\otimes})$. Since $\pi(b) \subset W^b_{\mathrm{NE}}$, $h^{\otimes}_i(b^{\otimes}) = h_i(b)$. By definition of W^b_{NE}, $\pi(b^{\otimes}_{-i}, b'_i) \subset W^b_{\mathrm{NE}}$. Hence, $h^{\otimes}_i(b^{\otimes}_{-i}, b'_i) = h_i(b^{\otimes}_{-i}, b'_i)$. Moreover, since (b_{-i}, b'_i) and $(b^{\otimes}_{-i}, b'_i)$ coincide in W^b_{NE}, $\pi(b_{-i}, b'_i) = \pi(b^{\otimes}_{-i}, b'_i)$. Hence, $h_i(b_{-i}, b'_i) = h_i(b^{\otimes}_{-i}, b'_i) > h_i(b^{\otimes}) = h_i(b)$. Contradicting the fact that $b \in \mathrm{NE}(G)$.

Second, we show that W^b_{NE} is a minimal closed and sufficient collection and thus, essential. By definition, $W^b_{\mathrm{NE}} = \langle W^b_{\mathrm{NE}} \rangle$. Let W be a sufficient and closed collection for NE, Γ, and b that does not contain W^b_{NE}. By Lemma 1, since W^b_{NE} is terminal, there is $\bar{z} \in (W^b_{\mathrm{NE}} \backslash W) \cap Z(\Gamma)$. Let $i \in N$ and $b'_i \in B(\Gamma)$ be such that $\bar{z} \in \pi(b_{-i}, b'_i)$. Consider the path of information sets from the root to $\bar{z}$, $\{u^1, \dots, u^k\}$, *i.e.*, $u^1 = r(\Gamma)$ and $u^k = \bar{z}$. Since W^b_{NE} is closed, $\{u^1, \dots, u^k\} \subset W^b_{\mathrm{NE}}$. Since W is also closed, $u^1 \in W$ and $u^k \notin W$, there is a unique $\bar{k}$ such that $u^{\bar{k}-1} \in W$ and

$u^{\bar{k}} \notin W$. Let $G = (\Gamma, h)$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, $h_i(z) = 0$. Let $\bar{G} = (\Gamma, \bar{h})$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, if $u^{\bar{k}} \preceq z$, $\bar{h}_i(z) := 1$ and $\bar{h}_i(z) := 0$ otherwise. Note that, since W is closed, $\bar{h}_i(z) = 1$ implies that $z \notin W$. Note that $b \in \mathrm{NE}(G)$ and $G \otimes_W \bar{G} = \bar{G}$. Since $\pi(b) \subset W$, in game $\bar{G}$, all the payoffs in $\pi(b)$ are 0. Take now $\hat{b} \in B(\Gamma)$ such that it coincides with b in W. Then, for each $i \in N$, $h_i(\hat{b}) = 0$. By construction, there is $z \in Z(\Gamma)$ such that $u^{\bar{k}} \preceq z$ and $z \in \pi(\hat{b}_{-i}, \bar{b}_i)$. Hence, $h_i(\hat{b}_{-i}, \bar{b}_i) > 0 = h_i(\hat{b})$, $\hat{b} \notin \mathrm{NE}(G \otimes_W \bar{G})$, contradicting the sufficiency of W. □

3.1.1 Nash equilibrium in strategic games

After the analysis above, it is natural to wonder what happens if we try to carry out a similar analysis for strategic games and also study the relations between the analysis for an extensive game and the associated strategic one. Figure 3 below shows that identifying the elements of a strategic game that are inessential to check if a given strategy profile is a NE is straightforward. The payoffs outside the shaded cross are not needed to check if (c_1, d_2) is a NE. More precisely, take an extensive form Γ, a strategy profile b and let $S(\Gamma)$ be the strategic form associated with Γ. Then, there is a one to one mapping between the *cells* in the cross associated with b in $S(\Gamma)$ and the terminal nodes of the essential collection $W_{\mathrm{NE}}(\Gamma, b)$. Yet, using strategic games to identify inessential elements for other equilibrium concepts like SPE or SE would become very cumbersome.

	a_2	b_2	c_2	d_2	e_2	f_2
a_1	6 , 5	2 , 3	8 , 2	1 , 3	1 , 4	2 , 5
b_1	6 , 5	1, 5	2 , 3	0 , 5	2 , 3	2 , 2
c_1	4 , 1	3 , 0	2 , 0	2 , 2	0 , 0	1 , 1
d_1	8 , 9	9 , 1	2 , 3	1 , 1	2 , 3	4 , 7

Figure 3: Essentializing NE in strategic games.

3.2 Subgame perfect equilibrium

Given $u \in U(\Gamma)$, let $W_u := \{v \in U(\Gamma) \,:\, u \preceq v\}$. A node $x \in X(\Gamma)$ is *elemental* if either it is a terminal node or, for each game (Γ, h), a *subgame* begins at x.[6] In particular, if x is elemental, then $u_x = \{x\}$. Given $x \in u \in U(\Gamma)$, let b_x and b_u denote the restriction of b to W_u. Consider the following definition of (nested) subsets of $U(\Gamma)$ (indeed, of elemental nodes).

Step 0**:** $X^0(b)$ coincides with the root of Γ.

Step t**:** An elemental node x belongs to $X^t(b)$ if there are $i \in N$, $b'_i \in B_i(\Gamma)$, and $y \in X^{t-1}(b)$ such that x is reached by $(b_{-i}, b'_i)_y$.

Then, let $X_{\mathrm{SPE}}(b) := \lim_{t\to\infty} X^t(b)$. Roughly speaking, $X_{\mathrm{SPE}}(b)$ consists of the elemental nodes that can be reached with a series of unilateral deviations from b. Since the game tree

[6]The notion of subgame we use is the standard one introduced in Selten (1975).

is finite, $X_{\text{SPE}}(b)$ is well defined. Let $W^b_{\text{SPE}} := \langle X_{\text{SPE}}(b)\rangle$. Note that W^b_{SPE} is a terminal collection.[7]

Proposition 2. *W^b_{SPE} is the essential collection for* SPE*, Γ, and b.*

Proof. First, we show that W^b_{SPE} is strongly sufficient for SPE, Γ, and b. Clearly, $\pi(b) \subset W^b_{\text{SPE}}$. Let $\bar{b} \in B(\Gamma)$ and $G = (\Gamma, h), \bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ be such that $b \in \text{SPE}(G)$ and $\bar{b} \in \text{SPE}(\bar{G})$. We show now that $b^{\otimes} \in \text{SPE}(G^{\otimes})$. Let $x \in X(\Gamma)$ be an elemental node. If $x \notin W^b_{\text{SPE}}$ then, since W^b_{SPE} is closed, $W_{u_x} \cap W^b_{\text{SPE}} = \emptyset$; hence, since $\bar{b} \in \text{SPE}(\bar{G})$, $b^{\otimes}$ induces a Nash equilibrium in the subgame of $G^{\otimes}$ that begins at x. If $x \in W^b_{\text{SPE}}$, by definition of W^b_{SPE}, no elemental node outside W^b_{SPE} can be reached with unilateral deviations from b at nodes in W^b_{SPE}. Hence, since $b \in \text{SPE}(G)$, $b^{\otimes}$ induces a Nash equilibrium in the subgame of $G^{\otimes}$ that begins at x. Hence, $b^{\otimes} \in \text{SPE}(G^{\otimes})$.

Second, we show that W^b_{SPE} is a minimal closed and sufficient collection and thus, essential. By definition, $W^b_{\text{SPE}} = \langle W^b_{\text{SPE}}\rangle$. Let W be a closed and sufficient collection for SPE, Γ and b that does not contain W^b_{SPE}. By Lemma 1, since W^b_{SPE} is terminal, there is $\bar{z} \in (W^b_{\text{SPE}} \backslash W) \cap Z(\Gamma)$. Consider the elemental nodes in the path from the root to $\bar{z}$, namely $\{x^1, \ldots, x^k\}$, where $x^1 = r(\Gamma)$ and $x^k = \bar{z}$. Since W^b_{SPE} is closed, $\{x^1, \ldots, x^k\} \subset W^b_{\text{SPE}}$. Since W is also closed, there is a unique $\bar{k} \geq 1$ such that $x^{\bar{k}-1} \in W$ and $x^{\bar{k}} \notin W$. Since W is sufficient, $\pi(b) \subset W$ and hence, $x^{\bar{k}} \in W^b_{\text{SPE}} \backslash \pi(b)$. Then, there are $i \in N$, $b'_i \in B_i(\Gamma)$ and $y \in X_{\text{SPE}}(b)$ such that $x^{\bar{k}}$ is reached by $(b_{-i}, b'_i)_y$ and not by b_y. Let $G = (\Gamma, h)$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, $h_i(z) = 0$. Let $\bar{G} = (\Gamma, \bar{h})$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, if $x^{\bar{k}} \preceq z$, $\bar{h}_i(z) := 1$ and $\bar{h}_i(z) := 0$ otherwise. Note that, since W is closed, $\bar{h}_i(z) = 1$ implies that $z \notin W$. Note that $b \in \text{SPE}(G)$ and $G \otimes_W \bar{G} = \bar{G}$. Take now $\hat{b} \in B(\Gamma)$ such that it coincides with b in W. Then, for each $i \in N$, $h_i(\hat{b}) = 0$. By construction, there is $z \in Z(\Gamma)$ such that $x^{\bar{k}} \preceq z$ that is reached by $(\hat{b}_{-i}, b'_i)_y$. Hence, in the subgame of $\bar{G}$ that begins at y, payoff 1 is obtained with positive probability instead of getting 0 for sure. Therefore, $\hat{b} \notin \text{SPE}(\bar{G}) = \text{SPE}(G \otimes_W \bar{G})$, contradicting the sufficiency of W. □

3.3 Perfect equilibrium

Given a game form Γ and a strategy profile $b \in B(\Gamma)$, the unique sufficient collection for PE, Γ, and b is $U(\Gamma)$. Therefore, $U(\Gamma)$ is the essential and essential collection for PE, regardless of the strategy profile b.

Proposition 3. *$U(\Gamma)$ is the essential collection for* PE*, Γ, and b.*

Proof. By definition, $U(\Gamma)$ is always closed, sufficient. Hence, it suffices to show that $U(\Gamma)$ is a minimal closed and sufficient collection and thus, essential. Let W be a closed and sufficient collection for SPE, Γ and b strictly contained in $U(\Gamma)$. By Lemma 1, since $U(\Gamma)$ is terminal, there is $\bar{z} \in (U(\Gamma) \backslash W) \cap Z(\Gamma)$ and, in particular, $\bar{z} \notin \pi(b)$. Let $G = (\Gamma, h)$ be such that, for each $i \in N$ and each $z \in Z(\Gamma)$, $h_i(z) = 0$. Let $\bar{G} = (\Gamma, \bar{h})$ be such that, for each $i \in N$ and $\bar{h}_i(\bar{z}) := 1$ and $\bar{h}_i(z) := 0$ otherwise. Note that $b \in \text{PE}(G)$ and $G \otimes_W \bar{G} = \bar{G}$. Note that $\bar{G}$ has a unique perfect equilibrium in which $\bar{z}$ is reached with probability 1. Hence, if $\hat{b} \in B(\Gamma)$ coincides with b in W, since $\bar{z} \notin \pi(b)$, then, $\hat{b} \notin \text{PE}(\bar{G}) = \text{PE}(G \otimes_W \bar{G})$, contradicting the sufficiency of W. □

[7]Note that $\langle X^1(b)\rangle = W^b_{\text{NE}}$.

For the discussed equilibrium concepts, the essential collections exhibit a feature that, *a priori*, seems quite natural. Namely, for each game form Γ and each $b \in B(\Gamma)$, $W_{\text{NE}}(\Gamma, b) \subset W_{\text{SPE}}(\Gamma, b) \subset W_{\text{PE}}(\Gamma, b)$. Therefore, we might think that, in general, if two equilibrium concepts EC^1 and EC^2 are such that, for each game G, $EC^1(G) \subset EC^2(G)$, then, for each game form Γ and each $b \in B(\Gamma)$, $W_{\text{EC}^2}(\Gamma, b) \subset W_{\text{EC}^1}(\Gamma, b)$. The results in the next section show that the latter claim is not true.

4 Essentializing belief-based equilibrium concepts

4.1 Belief-based equilibrium concepts. A first Approach

In this section we turn to some of the main concepts that have been studied for extensive games with imperfect information. Remarkably, our main result, Theorem 1 applies to a wide family of belief-based equilibrium concepts.

Following Kreps and Wilson (1982), given a game form Γ, a *system of beliefs over* $X(\Gamma)\backslash Z(\Gamma)$ is a function $\mu : X(\Gamma)\backslash Z(\Gamma) \to [0, 1]$ such that, for each $u \in U(\Gamma)$, $\sum_{x \in u} \mu(x) = 1$. An *assessment* is a pair (b, μ), where b is a behavior strategy profile and μ is a system of beliefs. Let $\mathcal{M}(\Gamma)$ denote the set of all beliefs that can be defined for Γ. Given $W \subset U(\Gamma)$ and $\mu, \bar{\mu} \in \mathcal{M}(\Gamma)$, let $\mu \otimes_W \bar{\mu} := (\mu_W, \bar{\mu}_{-W})$; when no confusion arises, we use $\mu^{\otimes}$. We use $h^{\mu}_{iu}(b)$ to denote i's expected utility conditional on information set u having been reached, that the probability of being at each node $x \in u$ is given by μ and that b is to be played thereafter.

Let Γ be a game form, (b, μ) an assessment and $b'_i \in B_i$. Let $u \in U_i$. We say that b'_i is a *best reply of player* i *against* (b, μ) *at* u if $h^{\mu}_{iu}(b_{-i}, b'_i) = \max_{b''_i \in B_i} h^{\mu}_{iu}(b_{-i}, b''_i)$. An assessment (b, μ) is *sequentially rational* if, for each $i \in N$ and each $u \in U_i$, b_i is a best reply of player i against (b, μ) at u. We adopt a standard abuse of language and, given an equilibrium concept EC defined for assessments, we write $b \in \text{EC}(G)$ to mean that there is $\mu \in \mathcal{M}(\Gamma)$ such that $(b, \mu) \in \text{EC}(G)$.

Let $G = (\Gamma, h)$ be an extensive game and (b, μ) an assessment. Consider the following definition of (nested) subsets of $U(\Gamma)$.

Step 0**:** $U^0 := \langle \pi(b) \rangle$.[8]

Step t**:** An information set $v \in U(\Gamma)$ belongs to V^t if there are $i \in N$, $b'_i \in B_i(\Gamma)$, and an information set $u \in U^{t-1} \cap U_i(\Gamma)$ such that v is reached with positive probability by $(b_{-i}, b'_i)_u$ when the probabilities of the nodes in u are given by μ. Let $U^t := \langle V^t \rangle$.

Let $W^{b,\mu} := \lim_{t \to \infty} U^t$. Since the game tree is finite, $W^{b,\mu}$ is well defined. Note that $W^{b,\mu}$ is a terminal collection. Figure 4 provides an example of the previous definition.

4.1.1 Weak perfect Bayesian equilibrium

Let $G = (\Gamma, h)$ be an extensive game. An assessment (b, μ) is *weakly consistent with Bayes rule* if μ is derived using Bayesian updating in the path of b. A *weak perfect Bayesian equilibrium* is an assessment that is sequentially rational and weakly consistent with Bayes rule.

Let Γ be a game form and let (b, μ) be an assessment that is *weakly consistent with Bayes rule*. We show below that, although $W^{b,\mu}$ is a natural candidate to be a sufficient collection

[8]For the sake of exposition, we do not make explicit the fact that the sets U^t and V^t depend on b and μ.

for WPBE, Γ, and b, something else is needed. Consider the game G in Figure 4. Consider the

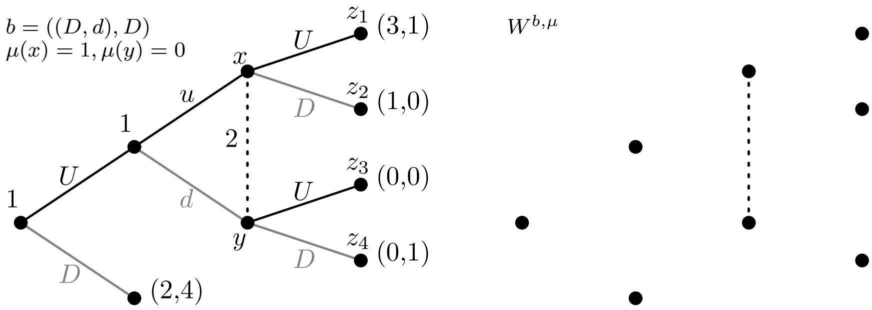


Figure 4: The collection $W^{b,\mu}$ is not sufficient.

assessments (b, μ) and $(b, \bar{\mu})$ where μ and $\bar{\mu}$ are such that $\mu(x) = 1$, $\mu(y) = 0$, $\bar{\mu}(x) = 0$, and $\bar{\mu}(y) = 1$. Since the information set of player 2 is off-path, all the beliefs in this game are weakly consistent with Bayes rule. Note that $(b, \mu) \notin \text{WPBE}(G)$ whereas $(b, \bar{\mu}) \in \text{WPBE}(G)$. Yet, to know that $b \in \text{WPBE}(G)$, it does not suffice to look at the payoffs in $W^{b,\mu}$. More formally, we show now that $W^{b,\mu}$ is not sufficient for WPBE, Γ, and b. In this example, $W^{b,\mu} = U(\Gamma)\backslash\{z_3\}$. Recall that $b \in \text{WPBE}(G)$. Let $\bar{G}$ be identical to G except for the fact that $\bar{h}(z_3) = (0, 2)$. Then, $\bar{b} = ((U, d), U) \in \text{WPBE}(\bar{G})$. Now, $b \otimes_{W^{b,\mu}} \bar{b} = b \notin \text{WPBE}(G \otimes_{W^{b,\mu}} \bar{G})$, since, in $G \otimes_{W^{b,\mu}} \bar{G}$, the choice D for player 2 is strictly dominated and hence, no beliefs make that choice sequentially rational. Similarly, by adequately rearranging the payoffs in the game G, it can be shown that $W^{b,\bar{\mu}}$ is not sufficient for WPBE, Γ, and b.

Despite of the discussion above, the collections $W^{b,\mu}$ are the key to essentialize WPBE. Let $\mathcal{M}^{\text{wc}}(b) := \{\mu \in \mathcal{M}(\Gamma) \;:\; (b, \mu) \text{ is weakly consistent with Bayes rule}\}$. Now, define the collection $W^b_{\text{WPBE}} := \bigcup_{\mu \in M^{\text{wc}}(b)} W^{b,\mu}$. Since the union of closed and terminal collections is a closed and terminal collection, W^b_{WPBE} is closed and terminal.

Proposition 4. W^b_{WPBE} *is the essential for* WPBE*,* Γ*, and* b*.*

Proof. This result is a particular case of the general result in Section 4.2 (Theorem 1). □

4.2 Belief-based equilibrium concepts. A general result

We develop now a general approach that allows to tackle several belief-based equilibrium concepts at once. Unfortunately, sequential equilibrium needs a separate treatment.

Let $\mathcal{F}$ be the set of all correspondences that select, for each game form Γ and each $b \in B(\Gamma)$, a subset of $\mathcal{M}(\Gamma)$ (the set of all beliefs that can be defined for Γ). [9]

Let Γ be a game form, $b \in B(\Gamma)$, and $G \in \mathcal{G}(\Gamma)$. Let $f \in \mathcal{F}$. We say that b is *sequentially rational under* f in game G, denoted by $b \in \text{SR}^f(G)$, if there is $\mu \in f(\Gamma, b)$ such that the assessment (b, μ) is sequentially rational. The above definition can be used to account for most belief-based solution concepts:

- Sequential rationality: $f^{\text{SR}}(\Gamma, b) := \mathcal{M}(\Gamma)$.

[9] More formally, let A denote the set of all pairs (Γ, b), where Γ is a game form and $b \in B(\Gamma)$. Then, $\mathcal{F} := \{\text{functions from } A \text{ to } 2^{\mathcal{M}(\Gamma)}\}$.

- WPBE: $f^{\text{WPBE}}(\Gamma, b) := \{\mu \in \mathcal{M}(\Gamma) \,:\, \mu \text{ is derived by Bayes rule in } \pi(b)\} = \mathcal{M}^{wc}(b)$.
- SE: $f^{\text{SE}}(\Gamma, b) := \{\mu \in \mathcal{M}(\Gamma) \,:\, \mu \text{ is consistent with } b\}$.
- Moreover, also the different versions of perfect Bayesian equilibrium that have been discussed in the literature can be defined as sequentially rational under some $f \in \mathcal{F}$.

Given $f \in \mathcal{F}$, a game form Γ, and $b \in B(\Gamma)$, define the collection $W^b_f := \cup_{\mu \in f(\Gamma,b)} W^{b,\mu}$. Note that, in particular, $W^b_{f^{\text{WPBE}}} = W^b_{\text{WPBE}}$. Since the union of closed and terminal collections is a closed and terminal collection, all the W^b_f collections are closed and terminal.

Lemma 4. *Let $f, f' \in \mathcal{F}$ be such that, for each Γ and each $b \in B(\Gamma)$, $f(\Gamma, b) \subset f'(\Gamma, b)$. Then, for each game G, $\text{SR}^f(G) \subset \text{SR}^{f'}(G)$.*

Proof. Straightforward. □

The next auxiliary lemma plays an important role in the proofs of the results in this section. Let $u, v \in U(\Gamma)$.

Lemma 5. *Let $f \in \mathcal{F}$. Let Γ be a game form and $b \in B(\Gamma)$. Let $W \subset U(\Gamma)$ be a closed collection containing $\pi(b)$ such that $W^b_f \backslash W \neq \emptyset$. Then, there are $i \in N$, $\tilde{u} \in W \cap U_i(\Gamma)$, $\tilde{v} \in W^b_f \backslash W$, $\tilde{\mu} \in f(\Gamma, b)$, $x_{\tilde{u}} \in \tilde{u}$, $x_{\tilde{v}} \in \tilde{v}$, and $\tilde{b}_i \in B_i(\Gamma)$ such that*

i) $x_{\tilde{u}} \prec x_{\tilde{v}}$ and $x_{\tilde{v}}$ (and hence, $\tilde{v}$) is reached with positive probability under $\tilde{\mu}$ by $(b_{-i}, \tilde{b}_i)_{\tilde{u}}$.

ii) Let $\{x^1 = x_{\tilde{u}}, \dots, x^l = x_{\tilde{v}}\}$ be the path from $x_{\tilde{u}}$ to $x_{\tilde{v}}$. For each $\bar{l} < l$, $u_{x^{\bar{l}}} \in W$.

Proof. By Lemma 1, there is $z \in (W^b_f \backslash W) \cap Z(\Gamma)$. Let $\tilde{\mu} \in f(\Gamma, b)$ be such that $z \in W^{b,\tilde{\mu}}$. Recall the (iterative) definition of $W^{b,\tilde{\mu}}$. Since $U^0 = \langle \pi(b) \rangle$ and $\pi(b) \subset W = \langle W \rangle$, then $U^0 \subset W$. Hence, there is $t \geq 1$ such that $z \in U^t \backslash U^{t-1}$. Let $u^t := z$. We now proceed backwards to identify the information sets used to reach u^t. Since $u^t \in U^t \backslash U^{t-1}$, there is $v^t \in V^t \backslash U^{t-1}$ such that $u^t \preceq v^t$ (indeed, since $u^t = z \in Z(\Gamma)$, in this first step $v^t = u^t$). Since $v^t \in V^t \backslash U^{t-1}$, there are $i^t \in N$, $b^t_{i^t} \in B_{i^t}(\Gamma)$, and $u^{t-1} \in (U^{t-1} \backslash U^{t-2}) \cap U_{i^t}(\Gamma)$,[10] such that v^t is reached with positive probability by $(b_{-i^t}, b^t_{i^t})_{u^{t-1}}$. Hence, we can define a sequence $\{u^0, v^1, u^1, \dots, v^t, u^t\}$, where $u^0 \in \langle \pi(b) \rangle$. Hence, $u^0 \in W$ and, since $u^t \notin W$, $W = \langle W \rangle$, and $u^t \preceq v^t$, we have that $v^t \notin W$; similarly, for each $t' \in \{0, \dots, t\}$, if $u^{t'} \notin W$, then $v^{t'} \notin W$. Let $\bar{t} := \min_{t' \in \{0,\dots,t\}} \{t' \,:\, u^{t'-1} \in W \text{ and } v^{t'} \notin W\}$. Define $i := i^{\bar{t}}$, $\tilde{u} := u^{\bar{t}-1}$, and $\tilde{b}_i := b^{\bar{t}}_{i^{\bar{t}}}$. Let $\bar{x} \in v^{\bar{t}}$ be such that $\bar{x}$ is reached with positive probability under $\tilde{\mu}$ by $(b_{-i}, \tilde{b}_i)_{\tilde{u}}$. Let $x_{\tilde{u}}$ be the node in $\tilde{u}$ such that $x_{\tilde{u}} \prec \bar{x}$. Let $\{\tilde{u} = w^0, w^1, \dots, w^k = v^{\bar{t}}\}$ be the path of information sets from $x_{\tilde{u}}$ to $\bar{x}$. All the information sets in $\{w^0, w^1, \dots, w^k\}$ are reached with positive probability under $\tilde{\mu}$ by $(b_{-i}, \tilde{b}_i)_{\tilde{u}}$. Since $w^0 \in W$, $w^k \notin W$, and $W = \langle W \rangle$, there is a unique $\bar{k}$ such that $w^{\bar{k}-1} \in W$ and $w^{\bar{k}} \notin W$. Now, define $\tilde{v} := w^{\bar{k}}$ and let $x_{\tilde{v}}$ be the node in the path from $x_{\tilde{u}}$ to $\bar{x}$ that belongs to $\tilde{v}$. So defined, it is clear that $\tilde{u} \in W \cap U_i$, $\tilde{v} \in W^{b,\tilde{\mu}}$ and hence, $\tilde{v} \in W^b_f \backslash W$; i) and ii) follow from the construction. □

For our general result we need to restrict to a subset of $\mathcal{F}$. Let $f \in \mathcal{F}$. We say f is *regular* if, given $b, \bar{b} \in B(\Gamma)$, the following properties hold

[10] If $t = 1$, then $U^{t-2} = U^{-1} := \emptyset$.

i) for each $\mu \in f(\Gamma, b)$ and each $\bar{\mu} \in f(\Gamma, \bar{b})$, $\mu \otimes_{W_f^b} \bar{\mu} \in f(\Gamma, b \otimes_{W_f^b} \bar{b})$ and, conversely,

ii) for each $\bar{\mu} \in f(\Gamma, b \otimes_{W_f^b} \bar{b})$, there is $\mu \in f(\Gamma, b)$ such that $\bar{\mu}$ and μ coincide in W_f^b.

In words, the beliefs inside W_f^b do not impose any restriction in the beliefs outside W_f^b and *vice versa*. According to the above definition, f^{SE} fails to be regular (see Example 5 in the Appendix) and hence, sequential equilibrium needs to be studied on his own.[11] Nonetheless, sequential rationality, WPBE, and many natural refinements of the latter can be defined through regular functions.[12]

Lemma 6. *Let $f \in \mathcal{F}$ be regular. If b and $\bar{b}$ coincide in W_f^b, then $W_f^b = W_f^{\bar{b}}$.*

Proof. Note that $\bar{b} = b \otimes_{W_f^b} \bar{b}$. We prove first that $W_f^b \subset W_f^{\bar{b}}$. Suppose, on the contrary, that there is $u \in W_f^b \backslash W_f^{\bar{b}}$. Take $i \in N$, $\tilde{u} \in W_f^{\bar{b}} \cap U_i$, $\tilde{v} \in W_f^b \backslash W_f^{\bar{b}}$, $\tilde{\mu} \in f(\Gamma, b)$, and $\tilde{b}_i \in B_i(\Gamma)$ as in Lemma 5. Since f is regular, there is $\bar{\mu} \in f(\Gamma, \bar{b})$ that coincides with $\tilde{\mu}$ in W_f^b. Since $b_{W_f^b} = \bar{b}_{W_f^b}$ and $W_f^b = \langle W_f^b \rangle$, for each $w \in U(\Gamma)$ such that $w \prec \tilde{v}$, $b_w = \bar{b}_w$ and hence, $\tilde{v}$ is reached with positive probability under $\bar{\mu}$ by $(\bar{b}_{-i}, \tilde{b}_i)_{\tilde{u}}$. Therefore, $\tilde{v} \in W_f^{\bar{b}}$ and we have a contradiction. Hence, $W_f^b \subset W_f^{\bar{b}}$.

We prove now that $W_f^{\bar{b}} \subset W_f^b$. Suppose, on the contrary, that there is $u \in W_f^{\bar{b}} \backslash W_f^b$. Take now $i \in N$, $\tilde{u} \in W_f^b \cap U_i$, $\tilde{v} \in W_f^{\bar{b}} \backslash W_f^b$, $\tilde{\mu} \in f(\Gamma, \bar{b})$, and $\tilde{b}_i \in B_i(\Gamma)$ as in Lemma 5. Since f is regular, there is $\mu \in f(\Gamma, b)$ that coincides with $\tilde{\mu}$ in W_f^b. If we had $b_{W_f^{\bar{b}}} = \bar{b}_{W_f^{\bar{b}}}$ we could follow as above. Yet, we just know that $b_{W_f^b} = \bar{b}_{W_f^b}$. From ii) in Lemma 5, all the information sets in the path from $x_{\tilde{u}}$ to $x_{\tilde{v}}$ belong to W_f^b. Hence, by i) in Lemma 5, if b and $\bar{b}$ coincide in W_f^b, $\tilde{v}$ is reached with positive probability under μ by $(b_{-i}, \tilde{b}_i)_{\tilde{u}}$ and we can derive the same contradiction as before. □

Theorem 1. *Let $f \in \mathcal{F}$ be regular. Then, W_f^b is the essential collection for SR^f, Γ, and b.*

Proof. First, we show that W_f^b is a strongly sufficient collection for SR^f, Γ, and b. By definition, $\pi(b) \subset W_f^b$. Let $\bar{b} \in B(\Gamma)$ and $G = (\Gamma, h)$, $\bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ be such that $b \in \mathrm{SR}^f(G)$ and $\bar{b} \in \mathrm{SR}^f(\bar{G})$. We claim that $(b^{\otimes}, \mu^{\otimes}) \in \mathrm{SR}^f(G^{\otimes})$. Since f is regular, $(b^{\otimes}, \mu^{\otimes}) \in f(\Gamma, b^{\otimes})$. We show now that it is sequentially rational. Let $u \in U(\Gamma)$. First, suppose that $u \notin W_f^b$. Since $W_f^b = \langle W_f^b \rangle$, for each $z \in Z(\Gamma)$ such that $u \prec z$, $z \notin W_f^b$ and hence, $h^{\otimes}(z) = \bar{h}(z)$. Therefore, since $\bar{b} \in \mathrm{SR}^f(\bar{G})$, $b^{\otimes}$ is sequentially rational at u in $G^{\otimes}$. So suppose $u \in W_f^b$. By definition of W_f^b, as far as beliefs in $f(W, b)$ are considered, no terminal node outside W_f^b is reached with positive probability after unilateral deviations from b at information sets in W_f^b; besides, by Lemma 6, $W_f^b = W_f^{b^{\otimes}}$ and hence, those terminal nodes are not reached either when the

[11]Moreover, also the perfect Bayesian equilibrium as defined in Fudenberg and Tirole (1991b) for multistage games with observed actions fails to be regular.

[12]For instance, Kreps and Wilson (1982) defined an equilibrium concept called *extended subgame perfect equilibrium*, a refinement of WPBE that imposes the use of Bayes rule off the equilibrium path (and hence, refines SPE as well). This equilibrium concept can be defined using regular functions.

beliefs in the information sets in W^b_f are taken from $f(W, b^{\otimes})$. Hence, since $b \in \mathrm{SR}^f(G)$, $b^{\otimes}$ is sequentially rational at u in $G^{\otimes}$. Hence, $b^{\otimes} \in \mathrm{SR}^f(G^{\otimes})$.

Second, we show that W^b_f is a minimal closed and sufficient collection and thus, essential. By definition, $W^b_f = \langle W^b_f \rangle$. Let W be a closed and sufficient collection for SR^f, Γ, and b that does not contain W^b_f. By Lemma 1, since W^b_f is terminal, there is $\bar{z} \in (W^b_f \backslash W) \cap Z(\Gamma)$. Let $\mu \in f(\Gamma, b)$ be such that $\bar{z} \in W^{b,\mu}$. Take $i \in N$, $\tilde{u} \in W \cap U_i$, $\tilde{v} \in W^b_f \backslash W$, $\tilde{\mu} \in f(\Gamma, b)$, $\tilde{b}_i \in B_i(\Gamma)$, $x_{\tilde{v}}$, and $x_{\tilde{u}}$ as in Lemma 5. Since $x_{\tilde{v}}$ is reached with positive probability under $\tilde{\mu}$ by $(b_{-i}, \tilde{b}_i)_{\tilde{u}}$, $\tilde{\mu}(x_{\tilde{u}}) > 0$.[13] Let $\tilde{c}$ denote the choice at $x_{\tilde{u}}$ that is in the path to $x_{\tilde{v}}$. We distinguish two cases:

Case 1: $b_i(\tilde{c}) = 0$, *i.e.*, according to b, choice $\tilde{c}$ is never made. Then, $\tilde{v}$ is not reached with positive probability under $\tilde{\mu}$ by $b_{\tilde{u}}$. Let $G = (\Gamma, h)$ be such that i) $(b, \tilde{\mu}) \in \mathrm{SR}^f(G)$ and ii) given a choice $c \neq \tilde{c}$ at $\bar{u}$, conditional on $\tilde{u}$ being reached, c is strictly dominated by $\tilde{c}$ in all nodes of $\tilde{u}$ but $x_{\tilde{u}}$. Since $\tilde{\mu}(x_{\tilde{u}}) > 0$, i) and ii) are compatible. Let $\bar{G} = (\Gamma, \bar{h})$ be such that, for each $j \in N$ and each $z \in Z(\Gamma)$, if $x_{\tilde{v}} \preceq z$, $\bar{h}_j(z) := M_G$ and $\bar{h}_j(z) := h_j(z)$ otherwise.[14] Since $\tilde{v} \notin W$ and $W = \langle W \rangle$, for each $z \in Z(\Gamma)$ such that $\tilde{v} \prec z$, $z \notin W$. Now, $b \in \mathrm{SR}^f(G)$ and $\mathrm{SR}^f(\bar{G}) \neq \emptyset$ (just take any strategy profile with payoff M_G). We claim that if $\hat{b} \in B(\Gamma)$ coincides with b in W, then $\hat{b} \notin \mathrm{SR}^f(G \otimes^W \bar{G})$. Note that $G \otimes^W \bar{G} = \bar{G}$. By construction, in game $\bar{G}$, conditional on $\tilde{u}$ being reached, $\tilde{c}$ is strictly dominant at $\tilde{u}$ (playing $\tilde{b}_i(\tilde{u})$ at $x_{\tilde{u}}$ leads to a payoff of M_G). Since $\tilde{u} \in W$, $b_i(\tilde{c}) = 0$ and $\hat{b}_W = b_W$, $\hat{b}$ is not sequentially rational at $\tilde{u}$.

Case 2: $b_i(\tilde{c}) > 0$. Now $\tilde{v}$ is reached with positive probability under $\tilde{\mu}$ by $b_{\tilde{u}}$. Let $G = (\Gamma, h)$ be such that i) $(b, \tilde{\mu}) \in \mathrm{SR}^f(G)$ and ii) there is a choice $c \neq \tilde{c}$ at $\bar{u}$ such that, conditional on $\tilde{u}$ being reached, c that strictly dominates $\tilde{c}$ in all nodes of $\tilde{u}$ but $x_{\tilde{u}}$. Let $\bar{G} = (\Gamma, \bar{h})$ be such that, for each $j \in N$ and each $z \in Z(\Gamma)$, if $x_{\tilde{v}} \preceq z$, $\bar{h}_j(z) := -(M_G)$ and $\bar{h}_j(z) := h_j(z)$ otherwise. The rest is very similar to Case 1. □

Corollary 2. *$U(\Gamma)$ is the essential collection for* SR, Γ*, and* b.

Proof. Immediate from Theorem 1 and the fact that $f^{\mathrm{SR}}(\Gamma, b) = \mathcal{M}(\Gamma)$. □

Corollary 3. *Let $f, f' \in \mathcal{F}$ be regular. Let Γ and $b \in B(\Gamma)$ be such that $f(\Gamma, b) \subset f'(\Gamma, b)$. Then, $W_{\mathrm{SR}^f}(\Gamma, b) \subset W_{\mathrm{SR}^{f'}}(\Gamma, b)$.*

Recall the claim we made at the end of Section 3. "If two equilibrium concepts EC^1 and EC^2 are such that, for each game G, $EC^1(G) \subset EC^2(G)$, then, for each game form Γ and each $b \in B(\Gamma)$, $W_{\mathrm{EC}^2}(\Gamma, b) \subset W_{\mathrm{EC}^1}(\Gamma, b)$". The above corollary implies that the claim is false (just think of SR and WPBE). Furthermore, when combined with Lemma 4, it also implies that, for belief-based equilibrium concepts, the opposite inclusion holds with a wide generality.

4.3 Decomposition of a game with respect to a collection

We introduce now a construction that is important to characterize the essential collections for sequential equilibrium in Section 4.4 and also for the analysis in Section 5.

[13] The arguments that begin now go through regardless of whether $\tilde{u}$ and $\tilde{v}$ a singletons and regardless of whether $\tilde{v}$ is a terminal node or not.

[14] Recall that $M_G := \max_{i \in N, z \in Z(\Gamma)} |h_i(z)| + 1$.

Let Γ be a game form and let $G = (\Gamma, h)$. Let $W \subset U(\Gamma)$ be a closed collection. We decompose G in a reduced game, G_W, and its complement; one containing the information sets in W and the other with those in $U(\Gamma)\backslash W$. Figure 5 illustrates the construction. Let $X(W) := \{x \in X(\Gamma) : u_x \in W\}$ and let $X(-W) := X(\Gamma)\backslash X(W)$. Let $A(W) := \{x \in X(-W) : y \prec x \Rightarrow y \in X(W)\}$, *i.e.*, $A(W)$ contains the nodes in $X(-W)$ with no predecessors in $X(-W)$.

We define the *reduced game* $G_W = (\Gamma_W, h_W)$, illustrated in Figure 5(c). We refer to Γ_W as the *reduced form* associated with Γ and W. Basically, the game form Γ_W is the restriction of Γ to $X(W)$. Nonetheless some artificial terminal nodes need to be added to ensure that we have a well defined game form.[15] Formally, $X(\Gamma_W) := X(W) \cup A(W)$, $U(\Gamma_W) := (U(\Gamma) \cap W) \cup A(W)$ and $Z(\Gamma_W) := (Z(\Gamma) \cap W) \cup A(W)$. All the other elements of Γ_W are defined by restricting Γ to $X(\Gamma_W)$ in the natural way. Let $M \in \mathbb{R}$ be some constant; M is fixed throughout the paper. Typically, we think of $M = M_G$, but the choice of the payoff for terminal nodes outside W is irrelevant for our analysis.[16]. Now, for each $z \in Z(\Gamma_W)$, $h_W(z) = h(z)$ if $z \in W$ and $h_W(z) = (M, \ldots, M)$ if $z \notin W$. We discuss the importance of the reduced games in Section 5.

Let $b \in B^0(\Gamma)$. For each $x \in A(W)$, let $p(x, b)$ denote the probability that x is reached given b and conditional on $X(-W)$ being reached. Now, we use b and the nodes in $X(-W)$ to define game $G(-W, b) = (\Gamma_{-W,b}, h_{-W,b})$; see Figure 5(d). The game form $\Gamma_{-W,b}$ is defined as follows. The root of $\Gamma_{-W,b}$ is a node $r_{-W} \notin X(\Gamma)$. $X(\Gamma_{-W,b}) := X(-W) \cup r_{-W}$. For each $x \in A(W)$, there is an arc from r_{-W} to x and the corresponding choice has probability $p(x, b)$. The rest of the elements are defined by restricting Γ to $X(\Gamma_{-W,b})$ in the natural way; in particular, $Z(\Gamma_{-W,b}) = X(-W) \cap Z(\Gamma)$ and, for each $z \in Z(\Gamma_{-W,b})$, $h_{-W,b}(z) = h(z)$. Note that, given $b, \bar{b} \in B^0(\Gamma)$, $G(-W, b)$ and $G(-W, \bar{b})$ only differ in the probabilities of nature move at the root. The games $G(W, b)$ are crucial to prove Proposition 5 below.

4.4 Sequential equilibrium

We say that a behavior strategy $b \in B$ is *completely mixed* if at each information set all the choices are taken with positive probability. Let B^0 denote the set of all completely mixed behavior strategy profiles. Let Γ be an extensive game. An assessment (b, μ) is *consistent* if there is some sequence $\{b_n\}_{n\in\mathbb{N}} \subset B^0$, such that $(b, \mu) = \lim_{n\to\infty}(b_n, \mu^{b_n})$, where μ^{b_n} denotes the unique beliefs that are consistent with Bayes rule given b_n. A *sequential equilibrium* is an assessment that is sequentially rational and consistent.

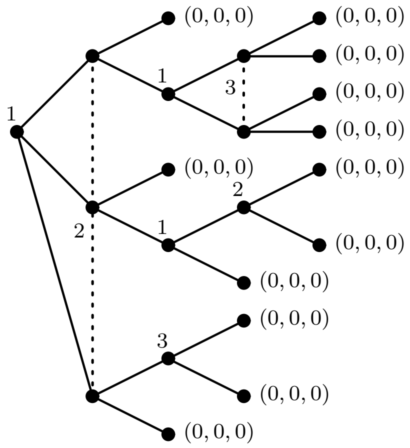
Let $\mathcal{M}^{\text{cons}}(b) := \{\mu \in \mathcal{M}(\Gamma) : (b, \mu) \text{ is consistent}\}$. Now, define the game form $W^b_{\text{SE}} := W^b_{f^{\text{SE}}} = \bigcup_{\mu\in\mathcal{M}^{\text{cons}}(b)} W^{b,\mu}$.

The fact that the W^b_{SE} combination of consistent assessments needs not be a consistent assessment implies that the function f^{SE} is not regular and hence, Theorem 1 can not be applied to SE. In Example 5 in the Appendix we illustrate why that f^{SE} is not regular and furthermore, that W^b_{SE} needs not be a strongly sufficient collection for sequential equilibrium.

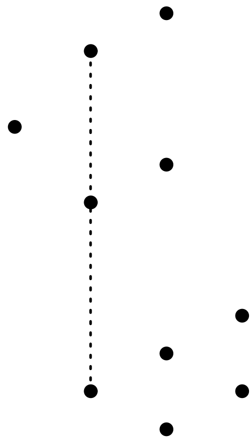
Proposition 5. *W^b_{SE} is the essential collection for* SE, Γ, *and* b.

[15] Consider the game in Figure 5. Suppose that we try to define a game form by restricting Γ to the nodes in $X(W)$ without adding any extra node. Then, in the information set of player 2 that contains three nodes, the number of choices available to player 2 would not be the same for the different nodes.

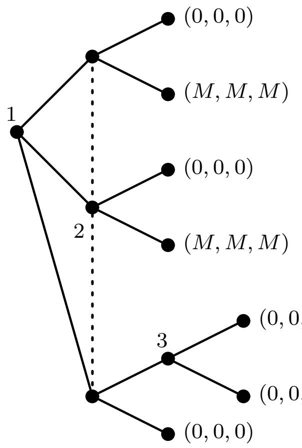
[16] Indeed, it is not even needed that the payoffs are equal across players or across terminal nodes (outside W), but it facilitates the exposition.



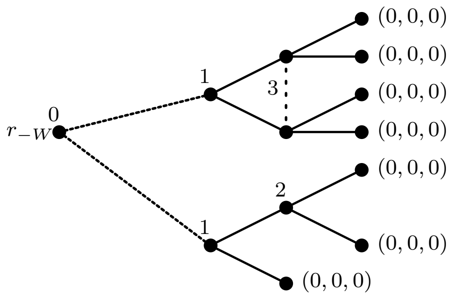
(a) The game G.



(b) A closed collection $W \subset U(\Gamma)$.



(c) The reduced game G_W.



(d) The (quasi) game $G(-W, \cdot)$.

Figure 5: Decomposition of a game with respect to a closed collection

Proof. First, we show that W^b_{SE} is a sufficient collection for SE, Γ, and b. By definition, $\pi(b) \subset W^b_{\mathrm{SE}}$. Let $G = (\Gamma, h), \bar{G} = (\Gamma, \bar{h}) \in \mathcal{G}(\Gamma)$ be such that $b \in \mathrm{SE}(G)$. We want to show that there is $\hat{b} \in \mathrm{SE}(G^{\otimes})$ that coincides with b in W^b_{SE}. Since $b \in \mathrm{SE}(G)$, there is $\mu \in \mathcal{M}^{\mathrm{cons}}$ such that (b, μ) is sequentially rational. Hence, there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ of completely mixed strategies converging to b such that the associated consistent beliefs, namely $\{\mu_n\}_{n\in\mathbb{N}}$, converge to μ.

We use now the games defined in Section 4.3. Consider the games $\{\bar{G}(-W^b_{\mathrm{SE}}, b_n)\}_{n\in\mathbb{N}}$. Let $n \in \mathbb{N}$ and let u be an information set of $\bar{G}(-W^b_{\mathrm{SE}}, b_n)$ formed by nodes in $A(W^b_{\mathrm{SE}})$. By definition, the beliefs induced by nature move at $r_{-W^b_{\mathrm{SE}}}$ in u coincide with μ_n. For each $n \in \mathbb{N}$, let $(\bar{b}_n, \bar{\mu}_n)$ be a sequential equilibrium of $\bar{G}(-W^b_{\mathrm{SE}}, b_n)$. The sequence $\{(\bar{b}_n, \bar{\mu}_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence; assume, without loss of generality, that the sequence itself converges and let $(\bar{b}, \bar{\mu})$ be its limit. We claim now that $b \otimes_{W^b_{\mathrm{SE}}} \bar{b} \in \mathrm{SE}(G^{\otimes})$. We show that $(b \otimes_{W^b_{\mathrm{SE}}} \bar{b}, \mu \otimes_{W^b_{\mathrm{SE}}} \bar{\mu})$ is a sequentially rational and consistent assessment.

Consistency: Let Γ_n be the game form of $\bar{G}(-W^b_{\mathrm{SE}}, b_n)$. By definition, for each $n, \bar{n} \in \mathbb{N}$, $B^0(\Gamma_n) = B^0(\Gamma_{\bar{n}})$. Let $\bar{B}^0 := B^0(\Gamma_n)$. Each $\bar{b}_n$ is a sequential equilibrium of $\bar{G}(-W^b_{\mathrm{SE}}, b_n)$. Hence, for each $n \in \mathbb{N}$, there is $\{\bar{b}_{n,k}\}_{k\in\mathbb{N}} \subset \bar{B}^0$ converging to b_n and such that associated beliefs (satisfying Bayes rule) converge to $\bar{\mu}_n$. Hence, for each $n \in \mathbb{N}$, there is $g(n) \in \mathbb{N}$ such that $\|\bar{b}_n - \bar{b}_{n,g(n)}\| \leq \frac{1}{n}$. Then, $\|\bar{b} - \bar{b}_{n,g(n)}\| \leq \|\bar{b} - \bar{b}_n\| + \|\bar{b}_n - \bar{b}_{n,g(n)}\| \leq \|\bar{b} - \bar{b}_n\| + \frac{1}{n}$. Hence, since $\bar{b}_n \to \bar{b}$, $\{\bar{b}_{n,g(n)}\}_{n\in\mathbb{N}} \to \bar{b}$. The convergence result for the corresponding beliefs, namely $\{\bar{\mu}_{n,g(n)}\}_{n\in\mathbb{N}}$, to $\bar{\mu}$ is analogous. Our construction ensures that, for each $n \in \mathbb{N}$ and each $x \in A(W^b_{\mathrm{SE}})$, $\mu_n(x) = \bar{\mu}_n(x)$ and $\mu(x) = \bar{\mu}(x)$, *i.e.*, the beliefs "match" in $A(W^b_{\mathrm{SE}})$. Hence, for each $n \in \mathbb{N}$, the beliefs associated with $b_n \otimes_{W^b_{\mathrm{SE}}} \bar{b}_{n,g(n)} \in B^0(\Gamma)$ are $\mu_n \otimes_{W^b_{\mathrm{SE}}} \bar{\mu}_{n,g(n)}$. Therefore, the consistency of $(b \otimes_{W^b_{\mathrm{SE}}} \bar{b}, \mu \otimes_{W^b_{\mathrm{SE}}} \bar{\mu})$ is obtained by considering the sequence $\{b_n \otimes_{W^b_{\mathrm{SE}}} \bar{b}_{n,g(n)}\}_{n\in\mathbb{N}}$.

Sequential rationality: The sequential rationality in the information sets in W^b_{SE} immediately follows from the sequential rationality of (b, μ) in G and the fact that, according to μ, no node outside W^b_{SE} can be reached with unilateral deviations from information sets W^b_{SE} and hence, the payoffs at all the terminal nodes that can be reached by unilateral deviations from information sets in W^b_{SE} are given by h. Similarly, only terminal nodes outside W^b_{SE} can be reached with unilateral deviations from information sets outside W^b_{SE} and hence, the payoffs are given by $\bar{h}$. Thus, since all the $\{(\bar{b}_n, \bar{\mu}_n)\}_{n\in\mathbb{N}}$ are sequentially rational also the limit, $(\bar{b}, \bar{\mu})$, is sequentially rational.

Second, since $W^b_{\mathrm{SE}} = W^b_{f^{\mathrm{SE}}}$, the proof the minimality is is analogous to the one for W^b_f in Theorem 1 (the regularity of f was not needed to show that W^b_f is minimally sufficient). ☐

It can be easily verified that $W_{\mathrm{SPE}} \subset W_{\mathrm{SE}}$. Hence, combining the results in Sections 3 and 4 we have:

$$W_{\mathrm{NE}} \subset W_{\mathrm{SPE}} \subset W_{\mathrm{SE}} \subset W_{\mathrm{WPBE}} \subset W_{\mathrm{SR}} = W_{\mathrm{PE}} = U.$$

5 The Reduced Game and its Applications

In this section we present some applications of the concepts of sufficient and essential collections. All of them are based on the reduced games defined in Section 4.3.

Proposition 6. *Fix an equilibrium concept* EC. *Let* Γ *be a game form and let* $b \in B(\Gamma)$. *Let* $G = (\Gamma, h) \in \mathcal{G}(\Gamma)$ *be such that* $\mathrm{EC}(G) \neq \emptyset$. *Let* W *be a closed collection sufficient (for* EC, Γ, *and* b*). Then, there is* $\hat{b} \in \mathrm{EC}(G)$ *such that* $\hat{b}_W = b_W$ *if and only if* $b_W \in \mathrm{EC}(G_W)$. *Moreover, since* $\pi(b) \subset W$, b *and* $\hat{b}$ *are realization equivalent.*

Proof. Suppose there is $\hat{b} \in \mathrm{EC}(G)$ such that $\hat{b}_W = b_W$. Let $\bar{G} \in \mathcal{G}(\Gamma)$ be a game with constant payoff $(M, \ldots, M)$. Since $b_W = \hat{b}_W$ and W is sufficient for EC, Γ, and b, then, by Lemma 2, W is sufficient for EC, Γ, and $\hat{b}$. Hence, there is $b^* \in \mathrm{EC}(G \otimes_W \bar{G})$ such that $b^*_W = \hat{b}_W$. Since, in game $G \otimes_W \bar{G}$, all the payoffs outside W coincide with $(M, \ldots, M)$, it is straightforward to check that $b_W \in \mathrm{EC}(G_W)$.

Suppose that $b_W \in \mathrm{EC}(G_W)$ and let $G^* = (\Gamma, h^*)$ be defined, for each $z \in Z(\Gamma) \cap W$, by $h^*(z) := h(z)$ and, for each $z \in Z(\Gamma) \backslash W$ and each $i \in N$, by $h^*_i(z) := M$. Since all the players are indifferent among the choices outside W and $b_W \in \mathrm{EC}(G_W)$, $b \in \mathrm{EC}(G^*)$. By definition, $G^* \otimes_W G = G$. Since W is sufficient (for EC, Γ, and b) and $\mathrm{EC}(G) \neq \emptyset$, there is $\hat{b} \in \mathrm{EC}(G)$ that coincides with b in W and moreover, since $\pi(b) \subset W$, $\hat{b}$ is realization equivalent to b. □

The above result provides a first application of sufficient collections. Given a strategy profile b and closed and sufficient collection W (for b), if b_W is an equilibrium of the reduced game, then the outcome of b is an equilibrium outcome in the original game. If b_W is not an equilibrium outcome of the reduced game, then no equilibrium of the original game will coincide with b in W. In particular, the reduced game associated to the essential collection would be the simplest of the games associated with b. Quite generally, the reduced game associated with an essential collection is notably simpler than the original game. Recall the discussion in the motivation section and refer to the games in Figure 7 below.

Apart from the immediate application described above, the reduced games can be also applied in different (though related) directions. In the remainder of this section we discuss two of them.

5.1 Structural robustness and partially-specified games

Let $G = (\Gamma, h)$ be a game form and let $W \in U(\Gamma)$ be a closed collection. Then, let $\Omega(W)$ denote the set of game forms such that if $\Lambda \in \Omega(W)$, then $W \subset U(\Lambda)$, W is closed in Λ and the nodes in W that are terminal in Γ are also terminal in Λ. Now, let $\mathcal{G}(W)$ denote the set of games $\hat{G} = (\Lambda, \hat{h})$ such that $\Lambda \in \Omega(W)$ and, for each $z \in W \cap Z(\Gamma)$, $\hat{h}(z) = h(z)$. Take, for example, the game G_W in Figure 5(c). Clearly, $G_W \in \mathcal{G}(W)$ and moreover, any other game that is defined from G_W by adding new branches at the nodes in $A(W)$ (those with payoff (M, M, M)) also belongs to $\mathcal{G}(W)$. These new branches can intersect each other, but cannot intersect W (since, otherwise, W would not be a closed collection in the resulting game form). In particular, the game G itself also belongs to $\mathcal{G}(W)$. We refer to the elements of $\mathcal{G}(W)$ as extensions of G_W.

Remark 2. Fix an equilibrium concept EC. Let Γ be a game form and let W be a closed collection. Let $\Lambda \in \Omega(W)$. Now, it is clear from the definitions of $\Omega(W)$ and Γ_W that $\Gamma_W = \Lambda_W$, *i.e.*, the corresponding reduced forms associated with W coincide.

In the remainder of this section, EC refers only to the equilibrium concepts whose essential collections we have characterized in Sections 3 and 4.

Proposition 7. *Fix an equilibrium concept* EC. *Let* Γ *be a game form and let* $b \in B(\Gamma)$. *Let* $\Lambda \in \Omega(W_{\mathrm{EC}}(\Gamma, b))$. *Let* $\bar{b} \in B(\Lambda)$ *be such that* b *and* $\bar{b}$ *coincide in* $W_{\mathrm{EC}}(\Gamma, b)$. *Then,* $W_{\mathrm{EC}}(\Gamma, b) = W_{\mathrm{EC}}(\Lambda, \bar{b})$, i.e., *the essential collections coincide.*

Proof. It can be shown that $W_{\mathrm{EC}}(\Gamma, b)$ is sufficient for EC, Λ, and $\bar{b}$ just by paralleling the arguments in the sufficiency part of the proof that $W_{\mathrm{EC}}(\Gamma, b)$ is sufficient for EC, Γ, and b but with Λ and $\bar{b}$ instead of Γ and b. Hence, $W_{\mathrm{EC}}(\Gamma, b) \subset W_{\mathrm{EC}}(\Lambda, \bar{b})$. Similarly, it can be shown that that $W_{\mathrm{EC}}(\Lambda, \bar{b})$ is sufficient for EC, Γ, and b and hence, $W_{\mathrm{EC}}(\Lambda, \bar{b}) \subset W_{\mathrm{EC}}(\Gamma, b)$. Therefore, $W_{\mathrm{EC}}(\Gamma, b) = W_{\mathrm{EC}}(\Lambda, \bar{b})$. □

Let W be the essential collection for EC, Γ, and b. Let $\bar{G}$ be an extension of G_W and let $\bar{b}$ be a strategy in $\bar{G}$ that coincides with b in W. Then, the next result shows that, to check if there is an equilibrium $\hat{b}$ of $\bar{G}$ that is realization equivalent to $\bar{b}$, then it suffices to check if $b \in \mathrm{EC}(G_W)$.

Corollary 4. *Fix an equilibrium concept* EC. *Let* $\mathcal{G}(W)$ *be the set of the extensions of a reduced game* $G_W = (\Gamma_W, h_W)$. *Let* $b \in B(\Gamma_W)$ *be such that* $W_{\mathrm{EC}}(\Gamma_W, b) = W$. *Let* $\bar{G} = (\Lambda, h) \in \mathcal{G}(W)$ *and* $\bar{b} \in B(\Lambda)$ *be such that* $b = \bar{b}_W$. *Then,*

i) $W_{\mathrm{EC}}(\Lambda, \bar{b}) = W$.

ii) If $b \in \mathrm{EC}(G_W)$, *then there is* $\hat{b} \in \mathrm{EC}(\bar{G})$ *such that* $\hat{b}_W = \bar{b}_W = b$.

iii) If $b \notin \mathrm{EC}(G_W)$, *then there is no* $\hat{b} \in \mathrm{EC}(\bar{G})$ *such that* $\hat{b}_W = \bar{b}_W = b$.

Proof. Statement i) is immediate from Proposition 7; ii) and iii) follow from Proposition 6. □

We describe now two applications of Corollary 4.

5.1.1 Structural robustness

We borrow the name of structural robustness from Kalai (2005, 2006), where similar changes in the underlying games are considered and used to study the robustness of Nash equilibria in large games.

Our approach allows us to study how robust the different equilibrium concepts are with respect to structural changes in the game. We already provided an illustration of this fact when comparing SE and WPBE in the licensing game (Section 2). Let EC be an equilibrium concept, $G = (\Gamma, h)$ an extensive game, and $b \in \mathrm{EC}(G)$. Suppose now that the game G is modified, by some changes in Γ or by some changes in h, and suppose that none of this changes affects the path of b. Let $\bar{G}$ be the modified game. Then, it is natural to ask whether the outcome of b is an equilibrium outcome for EC in $\bar{G}$ or not; essential collections are very useful here. Suppose that we have characterized the essential collections for EC. Then, if the changes in G affected neither $W_{\mathrm{EC}}(\Gamma, b)$ nor the payoffs in its terminal nodes, b is indeed an equilibrium outcome for EC in $\bar{G}$. To see this, just note that in the latter case, $\bar{G} \in \mathcal{G}(W_{\mathrm{EC}}(\Gamma, b))$, so Corollary 4 implies the desired result.

Note that the structural changes in the game can be of very different nature since they can: affect payoffs; change the sets of strategies; change the information available to the players; account for addition, elimination, or merging of players; enlarge or reduce the game; etc. As far as this changes do not affect the essential collection associated to a given equilibrium profile b, its outcome will be an equilibrium outcome also in the modified game; on the other hand, if the changes affected the essential collection, whether the outcome of b is remains an equilibrium outcome or not will depend on the specific payoffs of the games at hand.

Therefore, if the essential collections associated with an equilibrium concept EC^1 are always smaller than the ones associated with EC^2, we have that EC^1 is more robust to structural changes than it is EC^2. The latter statement, combined with the inclusion relations obtained for the essential collections characterized in sections 3 and 4 implies that SR and PE are the less robust equilibrium concepts followed, in this order, by WPBE, SE, SPE, and SR (the licensing exemplified this fact for SE and WPBE).

Remark 3. It is worth to provide one further clarification for what we mean when we say, for instance, that SE is structurally more robust than WPBE. Fix a game G and a strategy profile b. Suppose that b is a SE. The discussion above says that after any change in the game that does not affect the essential collection for SE, Γ, and b, the outcome of b remains a SE outcome in the modified game; no further calculation is needed, regardless of the actual payoffs of the modified game. On the other hand, suppose that b is just a WPBE but not a SE, then, since the latter changes might have affected the corresponding essential collection for WPBE (which is not smaller than the one for SE), b might not be a WPBE outcome anymore. Yet, our statement is mute about changes inside the essential collections. Indeed, since SE is more demanding than WPBE, it is natural to think that SE will be less robust to changes inside the essential collection.

5.1.2 Partially-specified games

As also discussed in Kalai (2005, 2006), the idea of structural robustness is very related to the possibility of dealing with partially-specified games. Let $G^p = (\Gamma, h)$ be a partially-specified game, *i.e.*, it lacks of a full description of Γ or some payoffs are unknown. Can we still say something about the equilibria of this game? Maybe. Suppose that there is a (possibly partially-specified) strategy $b \in B(\Gamma)$ such that $W_{\mathrm{EC}}(\Gamma, b)$ can be characterized and the corresponding reduced game is completely specified. Then, if $b_{W_{\mathrm{EC}}(\Gamma,b)} \in \mathrm{EC}(G^p_{W_{\mathrm{EC}}(\Gamma,b)})$, we know that, for whatever specification of the unknown elements of G^p, there is $\hat{b} \in \mathrm{EC}(G^p)$ that is realization equivalent to b, *i.e.*, the outcome of b will be an equilibrium outcome of any game satisfying the partial specifications of G^p.

A situation as the one described above may arise even in very simple settings. We have already mentioned one such situation when discussing the licensing game in Section 2. We present now an even simpler example.

Example 1. Consider the partially-specified game $G = (\Gamma, h)$ in Figure 6 below. We do not know how the game continues after x. It might be that x is a terminal node; it might be that we know the subgame beginning there, but that it is too complicated for its sequential equilibria to be found; and it might also be that we do not know anything at all about how the game follows once once x is reached. In any case, $W = \{u, v, z_1, z_2, z_3\}$ is the essential collection for SE, any such Γ and any strategy in which players 1 and 2 play D_1 and D_2 at their initial information sets. Hence, since $b = (D_1, D_2) \in \mathrm{SE}(G_W)$, there is a sequential equilibrium of G in which D_1 and D_2 are played, leading to the payoff vector $(1, 1)$. ◇

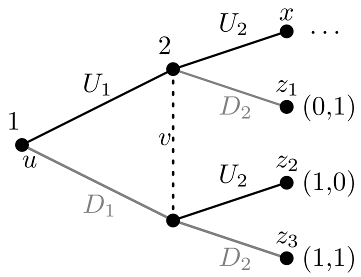


Figure 6: A partially-specified game

5.2 Virtual equilibrium concepts

All the analysis in the previous sections has been carried out in a framework in which the existence of the discussed equilibrium concepts was guaranteed. In this section we also allow for games with non-compact sets of strategies, discontinuous payoff functions, and also games in which only pure strategies are possible. Thus, there can be games without equilibria. Essentially, all the analysis and results in the previous sections carry over to these new settings although some care is needed.[17]

If a given equilibrium concept EC has been essentialized, then the *virtual version of* EC, VEC, can be defined as follows. Given $G = (\Gamma, h)$,

$$\mathrm{VEC}(G) := \{b \in B(\Gamma) \ : \ b_{W_{\mathrm{EC}}(\Gamma,b)} \in \mathrm{EC}(G_{W_{\mathrm{EC}}(\Gamma,b)})\}.$$

Clearly, for each game G, $\mathrm{EC}(G) \subset \mathrm{VEC}(G)$ and, by Proposition 6, if $\mathrm{EC}(G) \neq \emptyset$, then, for each $b \in \mathrm{VEC}(G)$, there is $\hat{b} \in \mathrm{EC}(G)$ realization equivalent to b.

Remark 4. The latter observation is the reason for the word "virtual". As far as the original game has some equilibrium, then the sets of equilibrium outcomes and virtual equilibrium outcomes coincide. Yet, there can be games in which the set of virtual equilibria is nonempty whereas there is no non-virtual equilibrium.

Remark 5. Virtual equilibria are very similar to the *trimmed equilibria* introduced in Groenert (2007). Yet, the analysis there only accounts for the trimmed versions of subgame perfect equilibrium and weak perfect equilibrium. At the same time, there are some differences in the two approaches. Roughly speaking, for each strategy profile b, we identify those information sets that are irrelevant to check if (for whatever beliefs) $b \in$ WPBE, *i.e.*, those information sets outside $\bigcup_{\mu \in M^{\mathrm{wc}}(b)} W^{b,\mu}$; on the other hand, in Groenert (2007), for each assessment (b, μ) with $\mu \in M^{\mathrm{wc}}(b)$, the author identifies those information sets that are irrelevant to check if $(b, \mu) \in$ WPBE, *i.e.*, those outside $W^{b,\mu}$.[18] Hence, the final equilibrium concepts, although stemming from the same ideas are slightly different.[19] We think that none of the two approaches outweighs the other; rather, they may be seen as complementary.

[17]For instance, in the proof of Proposition 5, it has to be ensured that $\mathrm{SE}(\bar{G}) \neq \emptyset$ implies that also the games $\bar{G}(-W^b_{\mathrm{SE}}, b_n)$ have some sequential equilibrium.

[18]Also, the analysis in Groenert (2007) just focuses on the definition of trimmed equilibria and there is no closedness requirement involved.

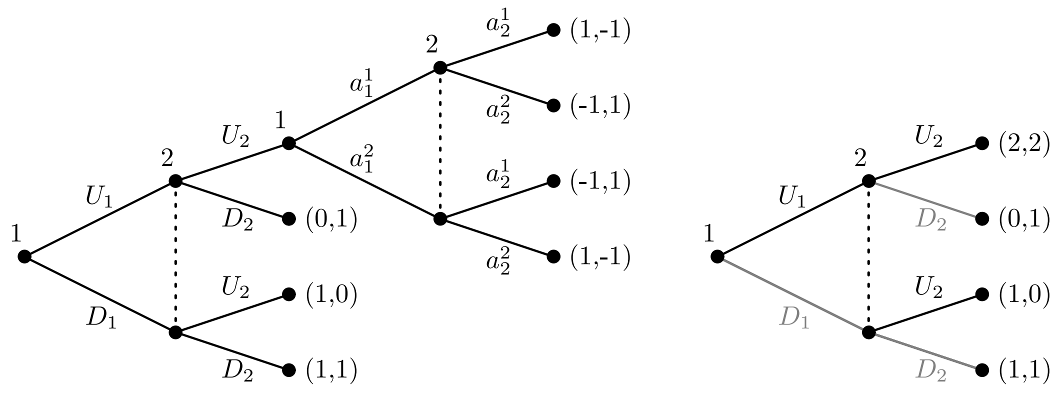
[19]The discussion above suggests that a virtual WPBE will always be a trimmed WPBE but there can be trimmed WPBE that are not virtual WPBE.

The following result says that the virtual versions of NE, SR, and PE coincide with the non-virtual versions. It is an immediate consequence of the corresponding characterizations of their essential collections.

Corollary 5. *For each game form* Γ *and each game* $G \in \mathcal{G}(\Gamma)$*,* $\mathrm{NE}(G) = \mathrm{VNE}(G)$*,* $\mathrm{SR}(G) = \mathrm{VSR}(G)$*, and* $\mathrm{PE}(G) = \mathrm{VPE}(G)$*.*

Nonetheless, for other equilibrium concepts, the virtual version can make a difference. Therefore, the virtual equilibrium concepts can lead to reasonable equilibrium behavior in settings where the classic equilibrium concepts fail to exist. We show this by elaborating on the situation we discussed in the motivation section.

Example 2. Consider the extensive game depicted in Figure 7. Suppose we restrict attention to pure strategies. Fix $k, l \in \{1, 2\}$ and let $b := \big((D_1, a_1^k), (D_2, a_2^l)\big)$. Note that in the subgame that begins after playing (U_1, U_2), there is no information set that belongs to $W_{\mathrm{SPE}}(\Gamma, b)$; the reduced game $G_{W_{\mathrm{SPE}}(\Gamma,b)}$ is depicted in Figure 7(b) (with $M = M_G$). Clearly, $b_{W_{\mathrm{SPE}}(\Gamma,b)} \in \mathrm{SPE}(G_{W_{\mathrm{SPE}}(\Gamma,b)})$ and hence, $b \in VSPE(G)$. However, if we restrict to pure strategies, $\mathrm{SPE}(G) = \emptyset$. We consider that, in the spirit of SPE, b is a sensible equilibrium of game G in the following sense. The players cannot use backwards induction to "solve" game G because the proper subgame does not have any NE. Still, suppose that the players are keen on backwards induction and insist on assigning payoffs at that subgame and then go backwards in the tree. Then, no matter what payoffs they assign to that subgame, they would find that b is a "solution" of the game. $\diamond$



(a) The game G. (b) The game $G_{W_{\mathrm{SPE}}(\Gamma,b)}$.

Figure 7: A game without SPE, but with VSPE.

In the example above, the game G did not have any SPE because we restricted attention to pure strategies. As we already illustrated in Section 2 with the licensing game, there may be other sources for the emptiness of the set of equilibria such as the discontinuity of the payoff functions (this was the case in the licensing game) or unboundedness of the payoffs. Moreover, the second version of the licensing game, LG^m, is a game with some VSE, but for which SE is not even defined.

Finally, we refer the reader to García-Jurado and González-Díaz (2006) for an application of the virtual subgame perfect equilibrium to derive a folk theorem in a repeated games setting in which the set of subgame perfect equilibria may be empty.

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A The closedness under $\preceq$

As we see in the proofs in Sections 3 and 4, although the intuitions behind the main results are quite simple, the proofs are somewhat cumbersome. In this respect, working with collections of informations sets that are closed under the precedence relation has proved to facilitate the analysis; not only for the results, but also for the applications discussed in Section 5. Nonetheless, as we argue below, even if we set aside the tractability issues, there are other reasons to require closedness in the definitions of essential collections.

In Example 3, we show that there can be reasonable sufficient collections that are not closed and that are strictly contained in the corresponding essential collections. Hence, the closedness assumption is not innocuous. In Example 4, we present a different situation in which unreasonable sufficient collections appear when the closedness requirement is removed; unreasonable in the sense that a (non-closed) sufficient collection might not contain informations sets that seem relevant for the equilibrium concept, game form, and strategy profile at hand.

Example 3. Consider a game form Γ that starts as depicted in Figure 8 and consider any strategy profile b in which player 1 plays D at $r(\Gamma)$ and player 2 plays u in his first (and possibly unique) information set. Then, by Proposition 1, $W_{\text{NE}}(\Gamma, b)$ consists of the closure of the information sets that can be reached after a unilateral deviation from b. It is easy to check that, although x cannot be reached by unilateral deviations, x belongs to $W_{\text{NE}}(\Gamma, b)$. Nonetheless, it seems that, given any game in $\mathcal{G}(\Gamma)$, x is not relevant to know if there is a NE that is realization equivalent to b. Therefore, it is arguable whether the essential collection for NE, Γ, and b should contain x or not. Note that this example cannot be trivially adapted, for instance, to SE, since the beliefs of player 4 might depend on the behavior at x and hence, adding x to an essential collection might be natural there. ◇

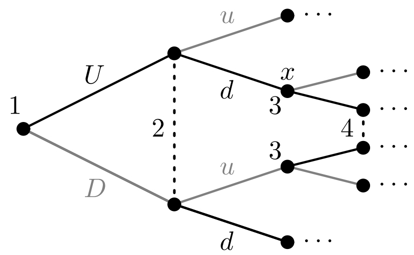


Figure 8: The closedness requirement is not inocuous.

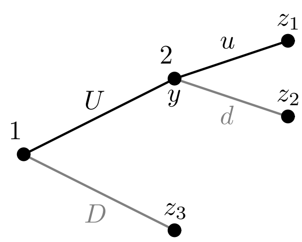


Figure 9: The closedness requirement can rule out unnatural collections.

Example 4. Consider now the game form Γ in Figure 9 and the strategy profile $b = (D, d)$. Clearly, $W_{\text{SPE}}(b, \Gamma) = \Gamma$, which is quite natural. On the other hand, it is easy to check that $W = \{r(\Gamma), z_1, z_2, z_3\}$ is a sufficient collection for SPE, Γ, and b. That is, node y is not needed, which is awkward. Differently from the situation in Example 3, the same analysis goes through for all the equilibrium concepts discussed in this paper. It is worth noting that the collection W is not strongly sufficient. Hence, we might strengthen sufficiency to strong sufficiency to rule out the above kind of collections. Yet, even if no other pathological examples appear when studying strong sufficiency without requiring closedness under $\preceq$, we know by Example 5 that strong sufficiency is very demanding for equilibrium concepts such as SE. ◇

B The licensing game

The example we present below does not pretend to be a real application of our analysis. It is an ad-hoc strategic situation that we use to illustrate the results in this paper and how they might be useful for game theoretical analysis. We have tried to keep it as simple as possible but, at the same time, rich enough to illustrate as many things as possible.

A government official (GO) has decided to grant a new telecommunications license and is considering to design an auction to allocate it; moreover, the control of GO over the overall process is limited by the present legislation and its options reduce to the ones we present below.[20] A local firm (LF) is interested in the auction and has valuation $v > 0$ for the license. A foreign firm (FF) is undecided among three choices: i) not entering the market, ii) enter with a high bid, and iii) enter with a low bid. At the same time, the government official is unsure about whether the presence of a foreign firm can be good for the interests of the country and has to decide whether to ban the entrance of the foreign firm or not. If FF does not enter the market, whatever the reason, the license goes to the local firm at price v; otherwise, there is an auction in which each participant has to pay an entry fee c, with $0 < c \ll v$. Moreover, only bides above $r > 0$ are allowed ($r \ll v$). The game runs as follows:

Stage 1: FF decides whether to enter or not in the market and whether to do it with a high bid $\bar{\alpha} = v$ or with a low bid $r < \underline{\alpha} \ll v - c$ (provided that its entrance is not banned by GO). FF's valuation of the license is slightly above $v + c$, say $v + 2c$.

Stage 1: Simultaneously and independently, GO decides whether to ban the entry of FF or not and, moreover, decides whether the auction will be simultaneous or sequential. In the latter case, LF would be informed about the bid of FF before making its own bid.

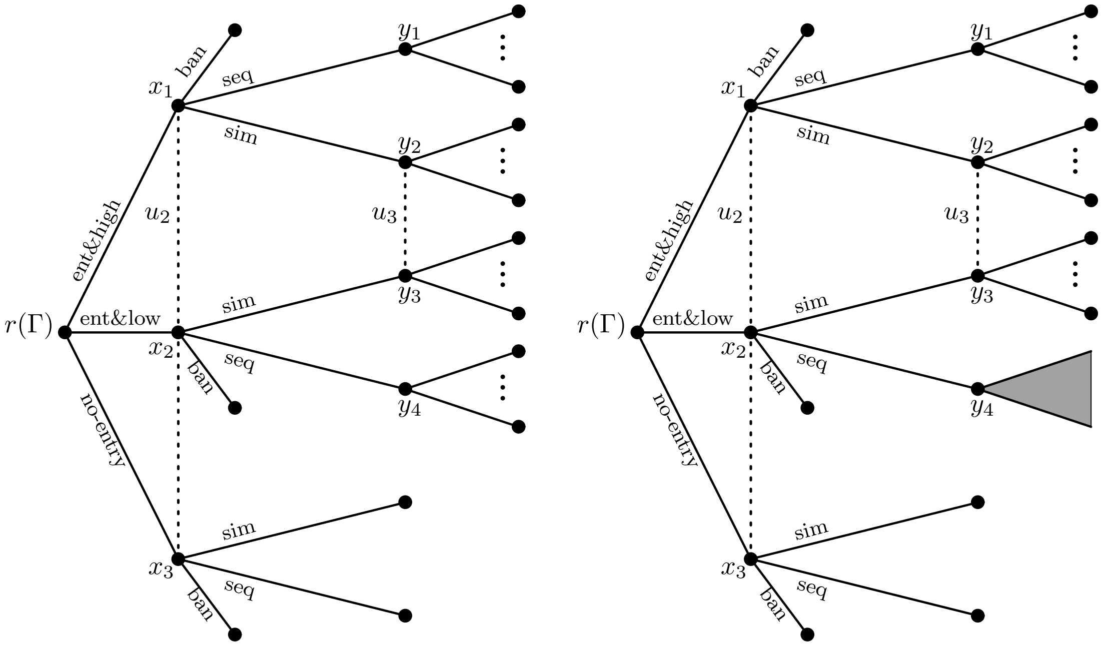
Stage 2: If after stage 1 FF is not in the market, then the license goes to LF at a price v. If FF is in the market, LF is informed about the action of GO and submits a bid. LF pays the entry cost only if his bid is positive (*i.e.*, a 0 bid is interpreted as not entering the auction).

End of the game: The license is allocated and the players pay their corresponding costs to GO. If both firms submit the same bid, the license is granted to each firm with equal probability. If the license goes to LF, then GO gets some extra utility given by $e > 2c$.

Moreover, we assume that the bids belong to a discrete set: there is a small number $\varepsilon > 0$ such that the only bids accepted are those of the form $k\varepsilon$, with $k \in \{0, 1, \ldots\}$ (r is assumed to be a valid bid). We denote this game by $\mathrm{LG}^d = (\Gamma^d, h^d)$, where d stands for "discrete bids". The extensive form associated with the game is depicted in Figure 10(a) (FF moves at $r(\Gamma)$, GO moves at u_2, and LF moves at y_1, u_3, and y_4).

We base our analysis in four profiles: the strategy profile $\bar{b}^1$ in which FF plays "ent&high" with probability 0.5 and "no-entry" with probability 0.5; GO plays "ban" with probability 1; and LF bids 0 at y_1 and $\underline{\alpha} + \varepsilon$ at u_3 and y_4. In this profile FF is best replying and, since $e > 2c$, GO is also best replying. LF is best replying at y_1 and y_4; and, if his beliefs at u_3 put high enough probability on y_3, then he is also best replying at u_3. Hence, since $u_3 \notin \pi(\bar{b}^1)$, the latter beliefs are compatible with the use Bayes rule in the path and therefore, $\bar{b}^1 \in \mathrm{WPBE}(\mathrm{LG}^d)$. Yet, the unique beliefs of LF at u_3 that are consistent (in the sense required by SE) put probability 1 on y_2. Hence, $\bar{b}^1 \notin \mathrm{SE}(\mathrm{LG}^d)$. The second strategy profile is $\bar{b}^2$, which only differs from $\bar{b}^1$ in the choice of LF at u_3, which is now bid 0. Clearly, if the beliefs of LF at u_3 put high enough probability on y_2, then he is best replying at u_3 when playing according to $\bar{b}^2$. Hence, $\bar{b}^2 \in \mathrm{SE}(\mathrm{LG}^d)$ and also $\bar{b}^2 \in \mathrm{WPBE}(\mathrm{LG}^d)$. Finally, b^1 and b^2 are defined from $\bar{b}^1$ and $\bar{b}^2$, respectively, by changing the bid of LF at y_4 to 0. Since bid 0 at y_4 is never sequentially rational, these new profiles are neither SE nor WPBE (but they still are NE).

[20] This game is just an example to illustrate the main results in this paper, *i.e.*, we do not pretend to provide a realistic model for the auctioning of telecommunication licenses.



(a) Γ^d, the game form of LG^d. (b) Γ^m, the game form of LG^m

Figure 10: The two licensing games.

Essential collections for WPBE and SE. Following the informal characterization above, the essential collection for WPBE, Γ^d, and b^1, namely W_{WPBE}, contains all the information sets of Γ^d with the exception of y_4 and the terminal nodes after y_4. To see why, just note that, since u_2 is in the path of b, any beliefs computed with Bayes rule in the path have to assign probability 0 at x_2; hence, according to any such beliefs, no deviation or series of deviations from b^1 reach y_4 or the terminal nodes after y_4 with positive probability (a simultaneous deviation by FF and GO would be needed). All the other information sets can be reached with (series) of unilateral deviations; for instance, to reach the terminal nodes that come after a bid β at u_3, consider beliefs that put probability 0.5 on y_2 and 0.5 on y_3 and consider the series of two deviations in which GO deviates to "sim" and, after observing this, LF deviates himself to bid β. On the other hand, the essential collection for SE, Γ^d, and b^1, namely W_{SE}, coincides with W_{WPBE} except for the fact it does not contain any of the terminal nodes that come after y_3. To see this, just recall that any beliefs consistent with b^1 will put probability 0 on y_3 and hence, the terminal nodes that come after y_3 are never reached with positive probability (according to any consistent beliefs) after any series of deviations from b^1. In particular, note that $W_{\mathrm{SE}} \subset W_{\mathrm{WPBE}}$. It can be easily seen that the essential collections for b^2, $\bar{b}^1$, and $\bar{b}^2$ are the same ones, W_{WPBE} for WPBE and W_{SE} for SE. Now we elaborate on what we can learn from essential collections.

The reduced game. In Section 5, given a game G, we associate a reduced game G_W with each (closed) collection of information sets W; the basic idea is to remove from G all the information sets that are not in W in such a way that what is left still forms a game. The reduced game $\mathrm{LG}^d_{W_{\mathrm{SE}}}$ would be defined as follows. Let $M \gg v$. We need to take care of y_4 and its successors and also of the successors of y_3. First, remove all the terminal nodes that come after y_4 and assign payoff (M, M, M) to y_4. We would like to do the same with y_3, but it belongs to u_3 and hence, to have a well defined reduced game, the same choices must be available to LF at y_2 and y_3. In this case, just replace all the payoffs of the terminal nodes that come after y_3 with the payoff (M, M, M). Then, in game $\mathrm{LG}^d_{W_{\mathrm{SE}}}$ we do not need to worry about the choices of LF at y_4 (which is now a terminal node) and, moreover, conditional on y_3 being reached, LF is indifferent between all his choices there. Given a strategy profile b, recall that b_W denotes the restriction of b to W, which is a strategy in the reduced game.

Given an equilibrium concept, a game, and a strategy profile, we can define a reduced game that helps to check if the outcome of the strategy profile is an equilibrium outcome. More specifically, suppose that we are given the strategy profile b^2 and we want to know if its outcome is a SE outcome of LG^d. Now, since W_{SE} is a sufficient collection (for SE, Γ^d, and b^2) and $b^2_{W_{\mathrm{SE}}}$ is a SE of the reduced game, (by Proposition 6) we have that, although $b^2 \notin \mathrm{SE}(\mathrm{LG}^d)$, its outcome is a SE outcome of LG^d.

Structural robustness. The main application of the reduced game may be to the study of the structural robustness of the different equilibrium concepts. We already know that $\bar{b}^1 \in \mathrm{WPBE}(\mathrm{LG}^d)$ and $\bar{b}^2 \in \mathrm{SE}(\mathrm{LG}^d)$ but, how robust are these equilibria to structural changes in the game? Suppose that, to reduce the advantage of LF and encourage the participation of FF, GO is considering the following changes in the way the license is granted, *i.e.*, changes in the game LG^d: **C1)** if "ent&low" and "seq" are played and FF looses the auction, then FF is given another chance to bid; **C2)** whenever "ent&low" has been played, FF is given another chance to bid if he looses the auction; **C3)** whenever FF looses the auction, FF is given another chance to bid. In this setting (by Corollary 4) each equilibrium is robust to changes outside its essential collection; so the equilibrium concepts with the smaller essential collections will be, to some extent, more robust. More precisely, even if **C1)** takes place, since this change occurs after y_4 (which does not belong to any of the essential collections), the outcome $\bar{b}^1$ would be a WPBE outcome of the modified game and the outcome of $\bar{b}^2$ a SE outcome, regardless of the specific details of **C1)**. Now, since the changes implied by **C2)** would come after y_3 and y_4, $\bar{b}^2$ would also be robust to **C2)**, but one should check again whether the outcome of $\bar{b}^1$ is a WPBE outcome of the reduced game, *i.e.*, SE is more robust to change **C2)** than WPBE. The latter feature holds in general, *i.e.*, since the essential collections associated with SE are smaller than the ones associated with WPBE; there are more changes after which there is nothing to reassess for SE than for WPBE (there are more changes that do not affect the essential collection of SE than the one for WPBE). Finally, change **C3)** affects both $\bar{b}^1$ and $\bar{b}^2$ and whether their outcomes remain equilibrium outcomes or not in the modified game would depend on the specific changes and our results are mute here. Indeed, in this last case, when the changes take place inside the essential collection, it is natural to expect that WPBE will be more robust than SE (since it is less demanding); actually, the outcome $\bar{b}^2$ might not be a SE outcome in the modified game and still be a WPBE outcome. Here we have just discussed three very simple modifications of the game, but the results hold for whatever changes are made outside the essential collections: appearance of new players, changes on the information partitions, changes in the payoffs, addition of subgames (no matter how big),. . .

Partial-specifications of the game. This issue is very related to the one above. The idea is that essential collections may help to give some information about the equilibrium outcomes of games that are not completely specified. Suppose that, in the licensing game, we have no idea about how the game continues once y_4 is reached. Even in this case we know (by Corollary 4) that, no matter how the game is defined from y_4 onwards, the outcome of $\bar{b}^2$ is going to be a SE outcome. Hence, essential collections help to identify what misspecifications in the game are irrelevant for different strategies and equilibrium concepts.

Virtual equilibrium concepts. Consider the following modification of the game LG^d. GO still has the same three actions but now, if he chooses "seq" after FF has played "ent&low", then LF can submit any real number above r as a bid (or 0); in the other cases the auctions are still over a discrete set. We denote this licensing game by LG^m, where m stands for mixed; see Figure 10(b). Suppose that we want to study the WPBE of LG^m. Then, since there is no best reply for LF at node y_4 (because of the discontinuity in the payoffs), $\mathrm{WPBE}(\mathrm{LG}^m) = \emptyset$. However, we consider that $\bar{b}^1$ is still as sensible in the game LG^m as it was in game LG^d; no matter the payoff that LF may get after y_4, that will not affect the sequential rationality of $\bar{b}^1$ at any other information set (provided that the beliefs are computed using Bayes rule in the path of $\bar{b}^1$). This motivates the definition of virtual equilibrium concepts. We say that a strategy profile b is a virtual WPBE if it is a WPBE of the reduced game associated with its essential collection (for WPBE and the game form at hand); and the virtual version of any other equilibrium concept is defined analogously. Now, $\bar{b}^1$ is a virtual WPBE of game LG^m. Note that also b^1 would be a virtual WPBE, despite of the irrational behavior at y_4. This is because virtual equilibrium concepts only impose restrictions in the behavior inside the essential collection. Given a virtual equilibrium, we can always replace the non-equilibrium behavior outside the essential collection by equilibrium behavior (if this exists) to get an equilibrium in the classic sense. Then, (by Proposition 6) if the set of WPBE of the original game is nonempty, the set of WPBE outcomes and virtual WPBE outcomes coincide (which justifies the name virtual). Furthermore, we can even have virtual equilibria in games in which the non-virtual counterpart is not defined. For instance, SE cannot be defined for games with uncountably many actions and hence, SE is not defined for game LG^m and, despite of this, since the sets of actions in the reduced game $\mathrm{LG}^d_{W_{\mathrm{SE}}}$ are again countable, we get that $\bar{b}^2$ is a virtual SE (*i.e.*, the nodes with uncountably many actions are in parts of the game that are irrelevant for $\bar{b}^2$).

C Strong sufficiency and sequential equilibrium

The example below shows that f^{SE} is not regular and that W^b_{SE} needs not be a strongly sufficient collection for sequential equilibrium.

Example 5. Consider the game $G \in \mathcal{G}(\Gamma)$ in Figure 11. Given $b = (D, D, (D, D))$, (b, μ) is a consistent assessment if and only if $\mu(a) = \mu(b) = 0$, $\mu(c) = 1$, $\mu(x) = \mu(\bar{x})$, and $\mu(y) = \mu(\bar{y})$. Now, W^b_{SE} is the collection that consists of removing the upper information set of player 3 and the four terminal nodes that come after it. More formally, $W^b_{\mathrm{SE}} = U(\Gamma)\backslash W$, where $W := \{v \in U(\Gamma) \,:\, u_x \preceq v\}$. Let $\mu \in \mathcal{M}^{\mathrm{cons}}(b)$ be such that $\mu(x) = \mu(\bar{x}) = 1$. So defined, $(b, \mu) \in \mathrm{SE}(G)$. Now, take the game $\bar{G} \in \mathcal{G}(\Gamma)$ depicted in the left part of Figure 12. Take $\bar{b} = b$. Let $\bar{\mu} \in \mathcal{M}^{\mathrm{cons}}(b) = \mathcal{M}^{\mathrm{cons}}(\bar{b})$ be such that $\bar{\mu}(x) = \bar{\mu}(\bar{x}) = 0$. So defined, $(\bar{b}, \bar{\mu}) \in \mathrm{SE}(\bar{G})$. Consider now the assessment $(b^{\otimes}, \mu^{\otimes})$. Since $b^{\otimes} = b$, $\mathcal{M}^{\mathrm{cons}}(b^{\otimes}) = \mathcal{M}^{\mathrm{cons}}(b)$. Therefore, since $\mu^{\otimes}(x) = 0 \neq 1 = \mu^{\otimes}(\bar{x})$, $\mu^{\otimes}$ is not consistent with $b^{\otimes}$. Hence, $(b^{\otimes}, \mu^{\otimes}) \notin \mathrm{SE}(G^{\otimes})$. Since $\mu \in \mathcal{M}^{\mathrm{cons}}(b) = f^{\mathrm{SE}}(\Gamma, b)$, $\bar{\mu} \in f^{\mathrm{SE}}(\Gamma, \bar{b})$, and $\mu^{\otimes} \notin f^{\mathrm{SE}}(\Gamma, b^{\otimes})$, we have shown that f^{SE} is not

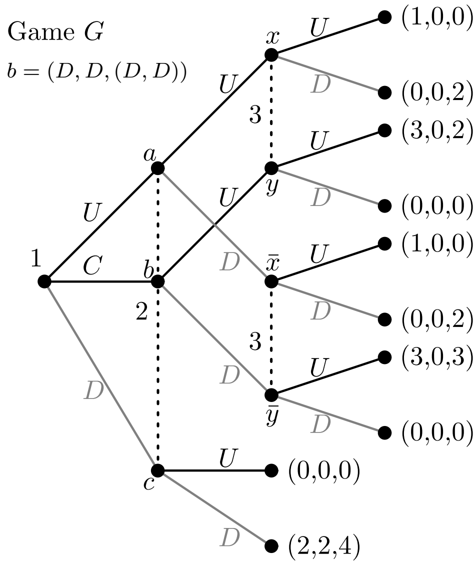


Figure 11: W^b_{SE} is not strongly sufficient for sequential equilibrium. The game G.

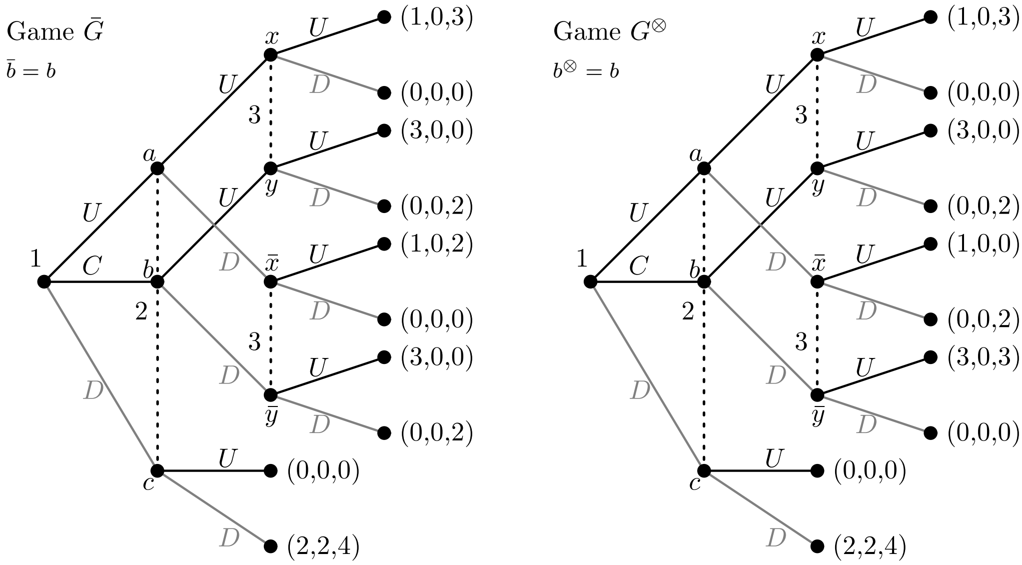


Figure 12: W^b_{SE} is not strongly sufficient for sequential equilibrium. The games $\bar{G}$ and $G^{\otimes}$.

regular. We show now that W^b_{SE} is not strongly sufficient for SE, Γ, and b by showing that $b^{\otimes} \notin \mathrm{SE}(G^{\otimes})$. Suppose, on the contrary, that $(b^{\otimes}, \hat{\mu}) \in \mathrm{SE}(G^{\otimes})$. Since player 3 is playing D at u_x and $(b^{\otimes}, \hat{\mu})$ is sequentially rational at u_x, $\hat{\mu}(y) \geq \frac{3}{5}$. Since $(b^{\otimes}, \hat{\mu})$ to be sequentially rational at $u_{\bar{x}}$, $\hat{\mu}(\bar{x}) \geq \frac{3}{5}$ and hence, $\bar{\mu}(\bar{y}) \leq \frac{2}{5}$. But this is not possible since, $\hat{\mu} \in \mathcal{M}^{\mathrm{cons}}(b^{\otimes})$ implies that $\bar{\mu}(y) = \bar{\mu}(\bar{y})$. $\diamond$